Cost-Partitioned Merge-and-Shrink Heuristics for Optimal Classical Planning: Technical Report

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This technical report contains a detailed proof for Theorem 7 of the paper *Cost-Partitioned Merge-and-Shrink Heuristics for Optimal Classical Planning* [1].

Theorem 1. *Given the FTS $F = \langle \Theta_1, \dots, \Theta_n \rangle$ with label set L and the FTS $F' = \langle \Theta'_1, \dots, \Theta'_n \rangle$ with label set L' that results from applying a label reduction $\lambda : L \to L'$ to F. Further, let ω be any order and ω' be derived from ω by replacing each Θ_i with Θ'_i.*

If λ is an exact label reduction, then $h^{\mathrm{SCP}}_{F,\omega} = h^{\mathrm{SCP}}_{F',\omega'}$.

Proof sketch. For the proof, we exploit the fact that all labels that are combined have to be Θ_k-combinable as the label reduction is exact and consider the order $\omega = \langle \Theta_1, \dots, \Theta_{k-1}, \Theta_k, \Theta_{k+1}, \dots, \Theta_n \rangle$ in the following.

We show by induction over ω that

$$h_{\Theta'_i}(s, rc'_{i-1}) = h_{\Theta_i}(s, rc_{i-1}) \text{ for } 1 \leq i \leq n \text{ and all } s. \tag{1}$$

The claim follows directly because the heuristic values under the SCP are equal to the heuristic values under the remaining cost function by definition of the SCP. We split the proof in three parts and show Equation 1 for the factors that are considered before Θ_k and Θ'_k in the computation of SCP in the first part, Θ_k and Θ'_k in the second part and the factors that are considered after Θ_k and Θ'_k in the computation of SCP in the third part.

In the first part, we show for the factors that are considered before Θ_k and Θ'_k that, in addition to Equation 1, it holds that

$$rc'_i(\ell') = rc_i(\ell) \text{ for } 0 \leq i < k, \text{ all } \ell' \in L' \text{ and all } \ell \in \lambda^{-1}(\ell') \tag{2}$$

$$cost'_i(\ell') = cost_i(\ell) \text{ for } 1 \leq i < k, \text{ all } \ell' \in L' \text{ and all } \ell \in \lambda^{-1}(\ell') \tag{3}$$

For the induction base $i = 0$, we have that $rc_0(\ell') = cost(\ell') = cost(\ell) = rc_0(\ell)$ for all $\ell \in \lambda^{-1}(\ell')$ because $rc_0 = cost$ in SCP and because exact label reduction requires that $cost(\ell_1) = cost(\ell_2)$ for all $\ell_1, \ell_2 \in \lambda^{-1}(\ell')$.

For the induction step from $i-1$ to i, it is important to consider the structure of the underlying factors. As all reduced labels are Θ_k-combinable, they may only have parallel transitions in all factors except Θ_k and hence also in $\Theta_1, \dots, \Theta_{k-1}$. Therefore, for all $\ell' \in L'$ and all $\ell_1, \dots, \ell_n \in \lambda^{-1}(\ell')$, the parallel transitions $s \xrightarrow{\ell_1} t, \dots, s \xrightarrow{\ell_n} t \in T_i$ are reduced to a transition $s \xrightarrow{\ell'} t \in T'_i$. We get that Θ_i and Θ'_i are graph-equivalent and therefore admit the same shortest paths. The proposition in Equation 1 follows from that given that Equations 1, 2 and 3 hold for $i - 1$.

With this, we can conclude that Equation 3 also holds for i, as the computation of saturated costs only depends on the transitions (which are equivalent for every $\ell \in L$ and for $\ell' \in L'$) and the identical heuristic values. Equation 2 holds in turn also for i as $rc_i = rc_{i-1} - cost_i$ and Equation 2 holds for $i-1$ due to the induction hypothesis and because Equation 3 holds for i.

In the second part, we consider the k-th factor. We have already shown that Equation 2 holds for the remaining cost functions rc'_{k-1} and rc_{k-1} that are used to compute the heuristic values in Θ'_k and Θ_k. Therefore, Equation 1 also holds for k.

However, it is no longer guaranteed that each pair of labels that is reduced to the same label only has parallel transitions, and Equations 2 and 3 no longer hold for that reason. Instead, it holds for all $\ell' \in L'$ that $cost'_k(\ell') = \max_{\ell \in \lambda^{-1}(\ell')} cost_k(\ell)$ because for all $\ell' \in L'$ we have that

$$\begin{aligned}
cost'_k(\ell') &= \max_{s \xrightarrow{\ell} t} h_{\Theta'_k}(s, rc'_{k-1}) - h_{\Theta'_k}(t, rc'_{k-1}) \\
&= \max_{s \xrightarrow{\ell} t} \bigcup_{\ell \in \lambda^{-1}(\ell')} h_{\Theta'_k}(s, rc'_{k-1}) - h_{\Theta'_k}(t, rc'_{k-1}) \\
&= \max_{s \xrightarrow{\ell} t} \bigcup_{\ell \in \lambda^{-1}(\ell')} h_{\Theta_k}(s, rc_{k-1}) - h_{\Theta_k}(t, rc_{k-1}) \\
&= \max_{s \xrightarrow{\ell} t} \max_{\ell \in \lambda^{-1}(\ell')} h_{\Theta_k}(s, rc_{k-1}) - h_{\Theta_k}(t, rc_{k-1}) \\
&= \max_{\ell \in \lambda^{-1}(\ell')} \max_{s \xrightarrow{\ell} t} h_{\Theta_k}(s, rc_{k-1}) - h_{\Theta_k}(t, rc_{k-1}) \\
&= \max_{\ell \in \lambda^{-1}(\ell')} cost_k(\ell)
\end{aligned}$$

With this and because Equation 2 holds for $k-1$, we get for all $\ell' \in L'$ that

$$\begin{aligned}
rc'_k(\ell') &= rc'_{k-1}(\ell') - cost'_k(\ell') \\
&= rc'_{k-1}(\ell') - \max_{\ell \in \lambda^{-1}(\ell')} cost_k(\ell) \\
&= \min_{\ell \in \lambda^{-1}(\ell')} rc'_{k-1}(\ell') - cost_k(\ell) \\
&= \min_{\ell \in \lambda^{-1}(\ell')} rc_{k-1}(\ell) - cost_k(\ell) \\
&= \min_{\ell \in \lambda^{-1}(\ell')} rc_k(\ell)
\end{aligned}$$

For the third part, we observe that the factors $\Theta_{k+1}, \ldots, \Theta_n$ and $\Theta'_{k+1}, \ldots, \Theta'_n$ share the same structural properties as the factors in the first part. We show Equation 1 by showing that Equation 3 and the following hold for the factors considered by SCP after Θ_k:

$$rc'_i(\ell') = \min_{\ell \in \lambda^{-1}(\ell')} rc_i(\ell) \text{ for } k \leq i \leq n \text{ and all } \ell' \in L' \tag{4}$$

For Equation 4, we have already shown that the induction base $i = k$ holds in the second part of this proof.

For the induction step from some $i-1$ to i, we observe that it is sufficient to only consider the cheapest transition among a set of parallel transitions in the computation of shortest paths and hence also in the computation of heuristic values. From graph-equivalence of Θ_i and Θ'_i and because of the induction hypothesis for the relationship between rc_{i-1} and rc'_{i-1} in Equation 4, it follows that Equation 1 holds for i.

It remains to show that the induction step holds for Equation 4. First observe that the induction step for Equation 3 holds for the same reasons as in the first part of this proof, and hence for all $\ell' \in L'$

$$\begin{aligned}
rc'_k(\ell') &= rc'_{k-1}(\ell') - cost'_k(\ell') \\
&= \min_{\ell \in \lambda^{-1}(\ell')} rc_{k-1}(\ell) - cost'_k(\ell') \\
&= \min_{\ell \in \lambda^{-1}(\ell')} rc_{k-1}(\ell) - cost_k(\ell) \\
&= \min_{\ell \in \lambda^{-1}(\ell')} rc_k(\ell)
\end{aligned}$$

□

References

[1] Silvan Sievers, Florian Pommerening, Thomas Keller, and Malte Helmert. Cost-partitioned merge-and-shrink heuristics for optimal classical planning. In *Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI 2020)*. IJCAI, 2020.