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Abstract
This technical report contains full proofs for the claim on redundant patterns for general cost partitioning of section “Restricting the Considered Patterns” of the main paper.

Definitions
We make some of the terminology used in the main paper formal definitions in the following.

Definition 1. Let \( \Pi \) be a planning task with variables \( V \), states \( S \), and cost function \( c \). A pattern \( P \) is redundant in \( \Pi \) for general cost partitioning if there are two patterns \( P_1, P_2 \subseteq P \) such that for every collection of patterns \( \mathcal{C} \subseteq 2^V \)

\[
h_{\mathcal{G}OCP}^{(P)}(s, c) = \min_{x \in \text{counts}(\mathcal{C}, s)} c(x)
\]

Definition 2. A pattern is interesting for a task for general cost partitioning if it is not redundant. It is interesting for a set of tasks if it is interesting in one of them.

Definition 3. A pattern \( P \) is causally connected if the subgraph of the causal graph induced by \( P \) is weakly connected. It is causally relevant if the subgraph of the causal graph induced by \( P \) contains a directed path via precondition edges from each node to some goal variable node.

Interesting Patterns for General Cost Partitioning
When talking about general cost partitioning, it is easier to consider its dual view which is operator counting. A flow in a graph is a function mapping edges to non-negative real numbers such that the total incoming and outgoing flow is balanced at each node (with exceptions for initial and goal nodes). The operator count of a flow maps each operator to the total flow along edges labeled with this operator. A pattern \( P \) is then associated with a set of operator counts for all valid flows in the abstract transition system of the projection to \( P \) after removing all dead states. Let us call this set \( \text{counts}(P, s) \). The cost of a count \( x \) under cost function \( c \) is simply \( c(x) = \sum_{o \in O} x(o)c(o) \). When considering a collection of patterns \( \mathcal{C} \), the operator-counting heuristic with the flow constraints for all patterns selects the cheapest operator count that has a valid flow in each projection, i.e., it considers the cheapest flow from the set \( \text{counts}(\mathcal{C}, s) = \bigcap_{P \in \mathcal{C}} \text{counts}(P, s) \). From the relation of general cost partitioning to operator counting, we know that

\[
h_{\mathcal{G}OCP}^{(P)}(s, c) = \min_{x \in \text{counts}(\mathcal{C}, s)} c(x)
\]

We can use this connection to show that \( h_{\mathcal{G}OCP}^{(P)}(s, c) \geq h_{\mathcal{G}OCP}^{(P')}(s, c) \) by showing that \( \text{counts}(\mathcal{C}, s) \subseteq \text{counts}(\mathcal{C}', s) \).

We show that the two conditions of Definition 1 of the main paper/Definition 3 of this technical report lead to redundant patterns for general cost partitioning in two separate theorems.

Theorem 1. A pattern \( P \) that is not causally connected is redundant for general cost partitioning.

Proof. Let \( P_1, P_2 \) be a partition of \( P \) into non-empty subsets such that the subgraph of the causal graph induced by \( P \) contains no arc between the two sets. Let \( \alpha, \alpha_1, \alpha_2 \) be the projections to \( P, P_1, P_2 \). It is sufficient to show that \( \text{counts}(P_1, s) \cap \text{counts}(P_2, s) \subseteq \text{counts}(P, s) \) for a state \( s \).

Consider a count \( x \in \text{counts}(P_1, s) \cap \text{counts}(P_2, s) \). There have to be flows \( f_1, f_2 \) in \( \alpha_1 \) and \( \alpha_2 \) with count \( x \). We define the following flow that can be seen as a way of concatenating abstract plans.

\[
f((\langle s, s' \rangle, o, \langle t, t' \rangle)) =
\begin{cases}
f_1((s, o, t)) & \text{if } s' = t' = \alpha_2(I) \\
w_s f_2((s', o, t')) & \text{if } s = t \text{ and } s' \neq t'
\end{cases}
\]

where \( w_s \) is the outgoing flow in \( s \) according to \( f_1 \). If there is a single goal state \( g \), then \( w_g = 1 \) and \( w_s = 0 \) for \( s \neq g \), otherwise the total outgoing flow of \( 1 \) can be split among all goal states.

As no operator can affect both \( P_1 \) and \( P_2 \), we can partition the operators into three sets \( O_1, O_2, O_3 \) of operators affecting \( P_1 \), operators affecting \( P_2 \) and operators affecting neither pattern.

Consider an operator \( o \in O_1 \cup O_3 \). It induces only self-loops in \( \alpha_2 \). For every transition \((s, o, t)\) in \( \alpha_1 \), there is exactly one transition \((s, \alpha_2(I)), o, (t, \alpha_2(I))\) in \( \alpha \) that has the same flow and all other transitions induced by \( o \) have a flow of 0. The count of \( o \) thus matches \( x(o) \).

Similarly, operators \( o \in O_2 \) match \( x(o) \) because each is counted with a weight of \( w_s \) for every state \( s \) in \( \alpha_1 \). These
weights have to sum to 1, so the values sum to the count of
o in f_2, which is x(o).

Finally, we have to check that f is a flow in α. When
considering only f_1, all states (s, s') with s' = α_2(I) have
an outgoing flow of w_{i}. The flow w_{i} f_2 then moves this flow
to a goal state while remaining within states with the first
component s.

We have shown that f is a flow in α with count x, so x ∈
counts(P). Since x was an arbitrary count in counts(P_1, s) ∩
counts(P_2, s), we have counts(P_1, s) ∩ counts(P_2, s) ⊆
counts(P, s).

Theorem 2. A pattern P is redundant for general cost parti-
tioning if it contains a variable v such that the causal graph
does not contain a directed path along precondition arcs
from v to a goal variable.

Proof. Let P be a pattern containing a variable v as de-
scribed above and let C be a collection of patterns. Further
let dep(v) be the set of variables than can be reached from
v via precondition-effect arcs in the causal graph. Note that
this set cannot contain a goal variable. Let O_{irr} be the subset
of all operators that have a precondition in dep(v). Operators
in O_{irr} cannot set goal variables nor any variables outside of
dep(v). Any plan (and in particular any abstract plan) is still
a plan once all of these operators are removed because the
operators reaching the goal conditions remain and all oper-
ators that remain can only have preconditions on variables
outside of dep(v), which were not modified by the removed
operators.

We first show that an optimal cost partitioning C =
⟨c_1, . . . , c_n⟩ for any pattern collection {P_1, . . . , P_n} re-
mains optimal if the cost of all operators in O_{irr} is changed
to 0 in all cost functions. To see this, consider a cheapest
abstract plan π in the projection to a pattern P_i under the
corresponding cost function c_i. The total contribution of op-

P is replaced by \{P', \emptyset\} because according to our defini-
tion a pattern is redundant if it can be replaced by two other
patterns.)