Discrete Structures

C2. Paths and Connectivity

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Walks, Paths, Tours and Cycles

Traversing Graphs

- When dealing with graphs, we are often not just interested in the neighbours, but also in the neighbours of neighbours, the neighbours of neighbours of neighbours, etc.
- Similarly, for digraphs we often want to follow longer chains of successors (or chains of predecessors).

Traversing Graphs

- When dealing with graphs, we are often not just interested in the neighbours, but also in the neighbours of neighbours, the neighbours of neighbours of neighbours, etc.
- Similarly, for digraphs we often want to follow longer chains of successors (or chains of predecessors).

Examples:

- circuits: follow predecessors of signals to identify possible causes of faulty signals
- pathfinding: follow edges/arcs to find paths
- control flow graphs: follow arcs to identify dead code
- computer networks: determine if part of the network is unreachable

Walks

Definition (Walk)

A walk of length n in a graph $\langle V, E \rangle$ is a tuple $\langle v_0, v_1, \ldots, v_n \rangle \in V^{n+1}$ s.t. $\{v_i, v_{i+1}\} \in E$ for all $0 \leq i < n$.

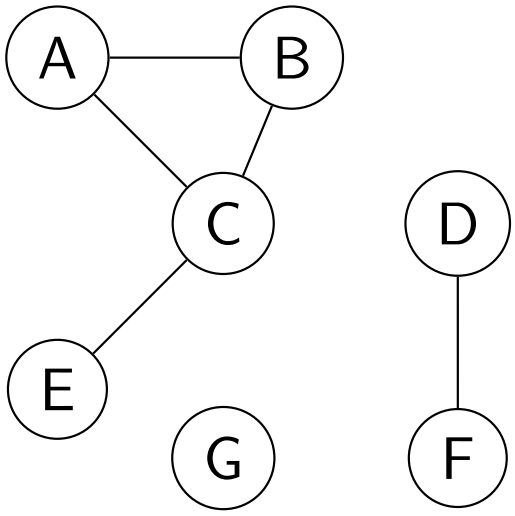
A walk of length n in a digraph $\langle N, A \rangle$ is a tuple $\langle v_0, v_1, \ldots, v_n \rangle \in N^{n+1}$ s.t. $\langle v_i, v_{i+1} \rangle \in A$ for all $0 \leq i < n$.

German: Wanderung

Notes:

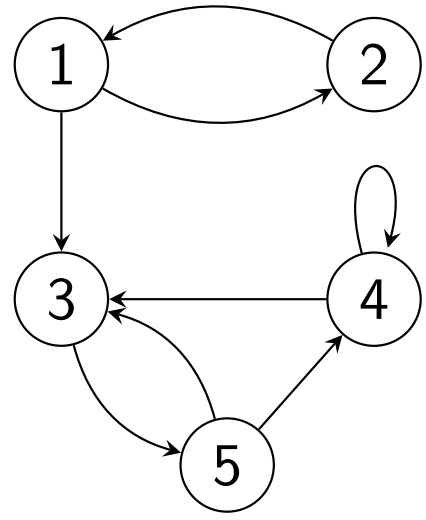
- The length of the walk does not equal the length of the tuple!
- The case $n = 0$ is allowed.
- Vertices may repeat along a walk.

Walks – Example



examples of walks:

- $\langle B, C, A \rangle$
- $\langle B, C, A, B \rangle$
- $\langle D, F, D \rangle$
- $\langle B, A, B, C, E \rangle$
- $\langle B \rangle$



examples of walks:

- $\langle 4, 4, 4, 4 \rangle$
- $\langle 3, 5, 3, 5 \rangle$
- $\langle 2, 1, 3 \rangle$
- $\langle 4 \rangle$
- $\langle 4, 4 \rangle$

Walks – Terminology

Definition

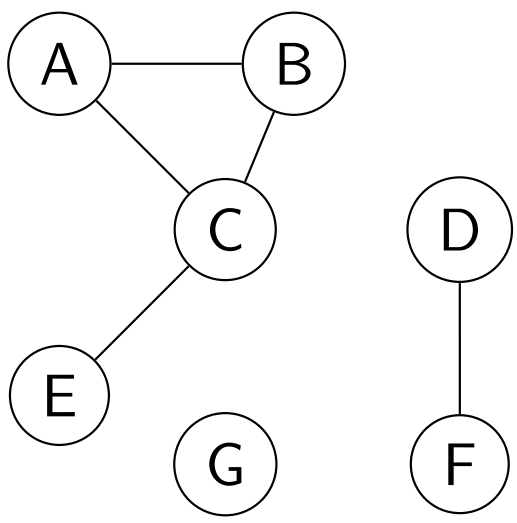
Let $\pi = \langle v_0, \ldots, v_n \rangle$ be a walk in a graph or digraph G.

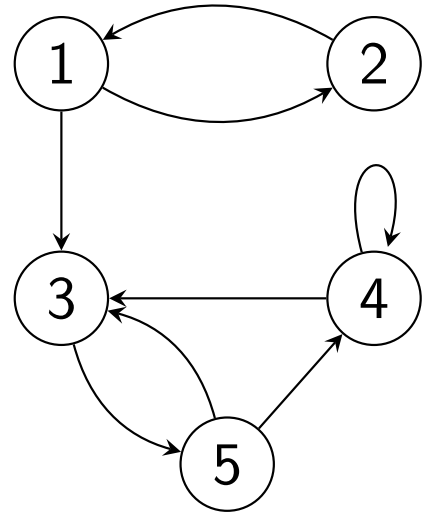
- We say π is a walk from v_0 to v_n.
- A walk with $v_i \neq v_j$ for all $0 \leq i < j \leq n$ is called a path.
- A walk of length 0 is called an empty walk/path.
- A walk with $v_0 = v_n$ is called a tour.
- A tour with $n \geq 1$ (digraphs) or $n \geq 3$ (graphs) and $v_i \neq v_j$ for all $1 \leq i < j \leq n$ is called a cycle.

German: von/nach, Pfad, leer, Tour, Zyklus

Note: Terminology is not very consistent in the literature.

Walks, Paths, Tours, Cycles – Example





Which walks are paths, tours, cycles?

- $\langle B, C, A \rangle$
- $\langle B, C, A, B \rangle$
- $\langle D, F, D \rangle$
- $\langle B, A, B, C, E \rangle$
- $\langle B \rangle$
- $\langle 4, 4, 4, 4 \rangle$
- $\langle 3, 5, 3, 5 \rangle$
- $\langle 2, 1, 3 \rangle$
- $\langle 4 \rangle$
- $\langle 4, 4 \rangle$

Reachability

Reachability

Definition (successor and reachability)

Let G be a graph (digraph).
The successor relation S_G and reachability relation R_G
are relations over the vertices/nodes of G defined as follows:

- $\langle u, v \rangle \in S_G$ iff $\{u, v\}$ is an edge ($\langle u, v \rangle$ is an arc) of G
- $\langle u, v \rangle \in R_G$ iff there exists a walk from u to v

If $\langle u, v \rangle \in R_G$, we say that v is reachable from u.

German: Nachfolger-/Erreichbarkeitsrelation, erreichbar

Reachability as Closure

Recall the n-fold composition R_n of a relation R over set S (Chapter B5):

- $R_0 = \{\langle x, x\rangle \mid x \in S\}$
- $R_n = R \circ R_{n-1}$ for $n \geq 1$

Theorem

Let G be a graph or digraph. Then:
$\langle u, v\rangle \in (\mathrm{S}_G)_n$ iff there exists a walk of length n from u to v.

Corollary

Let G be a graph or digraph. Then $\mathrm{R}_G = \bigcup_{n=0}^{\infty} (\mathrm{S}_G)_n$.

In other words, the reachability relation is the reflexive transitive closure of the successor relation.

Reachability as Closure – Proof (1)

Proof.

To simplify notation, we assume $G = \langle N, A \rangle$ is a digraph.
Graphs are analogous.
Proof by induction over n.

...

Reachability as Closure – Proof (1)

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To simplify notation, we assume $G = \langle N, A \rangle$ is a digraph.
Graphs are analogous.
Proof by induction over n.

induction base ($n = 0$):
By definition of the 0-fold composition, we have $\langle u, v \rangle \in (\mathrm{S}_G)_0$ iff $u = v$, and a walk of length 0 from u to v exists iff $u = v$.
Hence, the two conditions are equivalent. ...

Reachability as Closure – Proof (2)

Proof (continued).

induction step ($n \to n+1$):



Reachability as Closure – Proof (2)

Proof (continued).

induction step ($n \to n+1$):

($\Rightarrow$) : Let $\langle u, v\rangle \in (\mathsf{S}_G)_{n+1}$.
By definition of S_{n+1}, we get $\langle u, v\rangle \in \mathsf{S}_G \circ (\mathsf{S}_G)_n$.
By definition of $\circ$ there exists w with $\langle u, w\rangle \in (\mathsf{S}_G)_n$ and $\langle w, v\rangle \in \mathsf{S}_G$.
From the induction hypothesis, there exists a length-n walk $\langle x_0, \ldots, x_n\rangle$ with $x_0 = u$ and $x_n = w$.
Then $\langle x_0, \ldots, x_n, v\rangle$ is a length-$(n+1)$ walk from u to v.



Reachability as Closure – Proof (2)

Proof (continued).

induction step ($n \to n+1$):

($\Rightarrow$) : Let $\langle u, v\rangle \in (S_G)_{n+1}$.
By definition of S_{n+1}, we get $\langle u, v\rangle \in S_G \circ (S_G)_n$.
By definition of $\circ$ there exists w with $\langle u, w\rangle \in (S_G)_n$ and $\langle w, v\rangle \in S_G$.
From the induction hypothesis, there exists a length-n walk $\langle x_0, \ldots, x_n\rangle$ with $x_0 = u$ and $x_n = w$.
Then $\langle x_0, \ldots, x_n, v\rangle$ is a length-$(n+1)$ walk from u to v.

($\Leftarrow$) : Let $\langle x_0, \ldots, x_{n+1}\rangle$ be a length-$(n+1)$ walk from u to v ($x_0 = u$, $x_{n+1} = v$). Then $\langle x_n, x_{n+1}\rangle = \langle x_n, v\rangle \in A$.
Also, $\langle x_0, \ldots, x_n\rangle$ is a length-n walk from x_0 to x_n.
From the IH we get $\langle u, x_n\rangle = \langle x_0, x_n\rangle \in (S_G)_n$.
Together with $\langle x_n, v\rangle \in S_G$ this shows $\langle u, v\rangle \in S_G \circ (S_G)_n = (S_G)_{n+1}$.

□

Connected Components

Overview

- In this section, we study reachability of graphs in more depth.
- We show that it makes no difference whether we define reachability in terms of walks or paths, and that reachability in graphs is an equivalence relation.
- This leads to the connected components of a graph.
- In digraphs, reachability is not always an equivalence relation.
- However, we can define two variants of reachability that give rise to weakly or strongly connected components.

Walks vs. Paths

Theorem

Let G be a graph or digraph.
There exists a path from u to v iff there exists a walk from u to v.

In other words, there is a path from u to v iff v is reachable from u.



Walks vs. Paths

Theorem

Let G be a graph or digraph.
There exists a path from u to v iff there exists a walk from u to v.

In other words, there is a path from u to v iff v is reachable from u.

Proof.

($\Rightarrow$): obvious because paths are special cases of walks

($\Leftarrow$): Proof by contradiction. Assume there exist u, v such that there exists a walk from u to v, but no path. Let $\pi = \langle w_0, \ldots, w_n \rangle$ be such a counterexample walk of minimal length.
Because π is not a path, some vertex/node must repeat.
Select i and j with $i < j$ and $w_i = w_j$.
Then $\pi' = \langle w_0, \ldots, w_i, w_{j+1}, \ldots, w_n \rangle$ also is a walk from u to v.
If π' is a path, we have a contradiction.
If not, it is a shorter counterexample: also a contradiction. $\square$

Reachability in Graphs is an Equivalence Relation

Theorem

For every graph G, the reachability relation R_G is an equivalence relation.

In directed graphs, this result does not hold (easy to see).

Proof.

We already know reachability is reflexive and transitive.
To prove symmetry:

$$
\begin{aligned}
& \langle u, v \rangle \in \mathrm{R}_G \\
\Rightarrow\ & \text{there is a walk } \langle w_0, \ldots, w_n \rangle \text{ from } u \text{ to } v \\
\Rightarrow\ & \langle w_n, \ldots, w_0 \rangle \text{ is a walk from } v \text{ to } u \\
\Rightarrow\ & \langle v, u \rangle \in \mathrm{R}_G
\end{aligned}
$$



Connected Components

Definition (connected components, connected)

In a graph G, the equivalence classes
of the reachability relation of G
are called the connected components of G.

A graph is called connected if it has at most 1
connected component.

German: Zusammenhangskomponenten, zusammenhängend

Remark: The graph $\langle \emptyset, \emptyset \rangle$ has 0 connected components.
It is the only such graph.

Weakly Connected Components

Definition (weakly connected components, weakly connected)

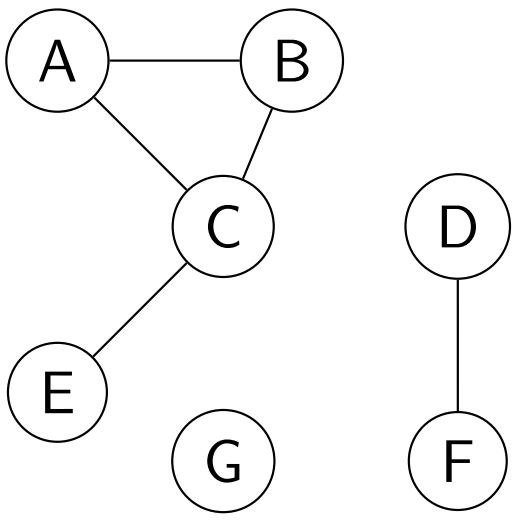
In a digraph G, the equivalence classes
of the reachability relation of the induced graph of G
are called the weakly connected components of G.

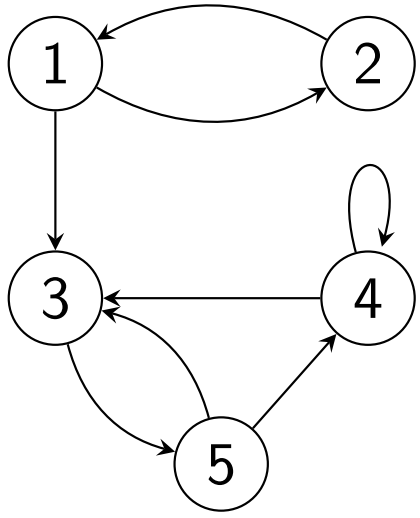
A digraph is called weakly connected if it has at most 1
weakly connected component.

German: schwache Zshk., schwach zusammenhängend

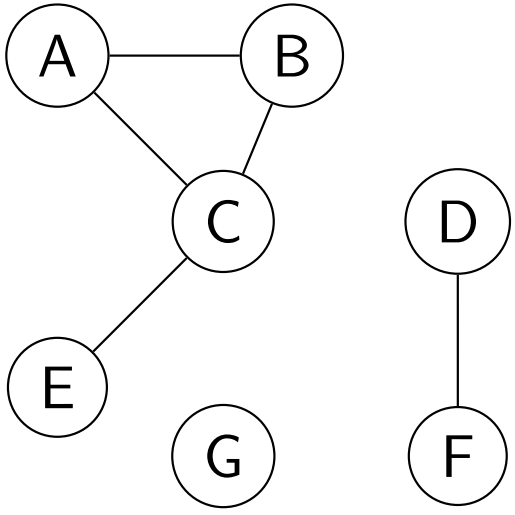
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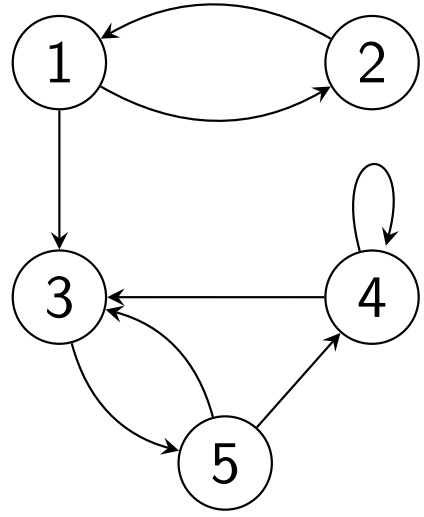
(Weakly) Connected Components – Example





(Weakly) Connected Components – Example





connected components:

- $\{A, B, C, E\}$
- $\{D, F\}$
- $\{G\}$

weakly connected components:

- $\{1, 2, 3, 4, 5\}$

Mutual Reachability

Definition (mutually reachable)

Let G be a graph or digraph.
Vertices/nodes u and v in G are called mutually reachable
if v is reachable from u and u is reachable from v.
We write M_G for the mutual reachability relation of G

German: gegenseitig erreichbar

Note: In graphs, $\mathrm{M}_G = \mathrm{R}_G$. (Why?)

Mutual Reachability is an Equivalence Relation

Theorem

For every digraph G, the mutual reachability relation M_G is an equivalence relation.

Mutual Reachability is an Equivalence Relation

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For every digraph G, the mutual reachability relation M_G is an equivalence relation.

Proof.

Note that $\langle u, v \rangle \in \mathrm{M}_G$ iff $\langle u, v \rangle \in \mathrm{R}_G$ and $\langle v, u \rangle \in \mathrm{R}_G$.

- reflexivity: for all v, we have $\langle v, v \rangle \in \mathrm{M}_G$ because $\langle v, v \rangle \in \mathrm{R}_G$

Mutual Reachability is an Equivalence Relation

Theorem

For every digraph G, the mutual reachability relation M_G is an equivalence relation.

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- reflexivity: for all v, we have $\langle v, v\rangle \in \mathrm{M}_G$ because $\langle v, v\rangle \in \mathrm{R}_G$
- symmetry: Let $\langle u, v\rangle \in \mathrm{M}_G$. Then $\langle v, u\rangle \in \mathrm{M}_G$ is obvious.



Mutual Reachability is an Equivalence Relation

Theorem

For every digraph G, the mutual reachability relation M_G is an equivalence relation.

Proof.

Note that $\langle u, v\rangle \in \mathrm{M}_G$ iff $\langle u, v\rangle \in \mathrm{R}_G$ and $\langle v, u\rangle \in \mathrm{R}_G$.

- reflexivity: for all v, we have $\langle v, v\rangle \in \mathrm{M}_G$ because $\langle v, v\rangle \in \mathrm{R}_G$
- symmetry: Let $\langle u, v\rangle \in \mathrm{M}_G$. Then $\langle v, u\rangle \in \mathrm{M}_G$ is obvious.
- transitivity: Let $\langle u, v\rangle \in \mathrm{M}_G$ and $\langle v, w\rangle \in \mathrm{M}_G$.
  Then: $\langle u, v\rangle \in \mathrm{R}_G$, $\langle v, u\rangle \in \mathrm{R}_G$, $\langle v, w\rangle \in \mathrm{R}_G$, $\langle w, v\rangle \in \mathrm{R}_G$.
  Transitivity of R_G yields $\langle u, w\rangle \in \mathrm{R}_G$ and $\langle w, u\rangle \in \mathrm{R}_G$,
  and hence $\langle u, w\rangle \in \mathrm{M}_G$.



Strongly Connected Components

Definition (strongly connected components, strongly connected)

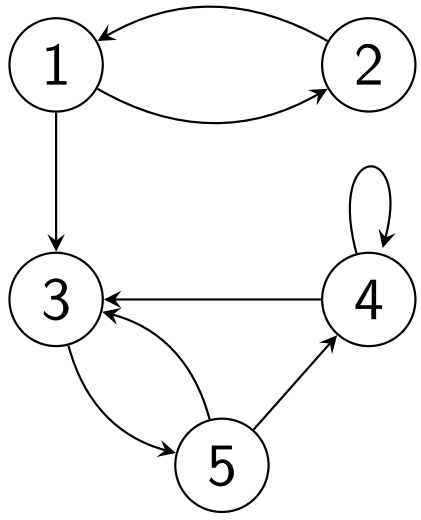
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A digraph is called strongly connected if it has at most 1
strongly connected component.

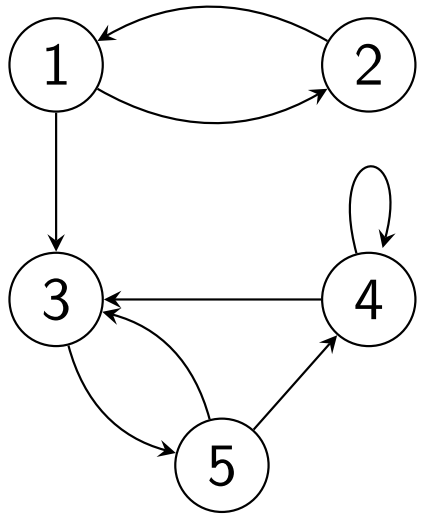
German: starke Zshk., stark zusammenhängend

Remark: The digraph $\langle \emptyset, \emptyset \rangle$ has 0 strongly connected components.
It is the only such digraph.

Strongly Connected Components – Example



Strongly Connected Components – Example



strongly connected components:

- $\{1, 2\}$
- $\{3, 4, 5\}$