Discrete Structures

C1. Introduction to Graphs

Malte Helmert, Gabriele Röger

University of Basel

November 2, 2026

C1.1 Graphs and Directed Graphs

C1.2 Induced Graphs and Degree Lemma

C1.1 Graphs and Directed Graphs

Graphs

Graphs (of various kinds) are ubiquitous in Computer Science and its applications.

Some examples:

- Boolean circuits in hardware design
- control flow graphs in compilers
- pathfinding in video games
- computer networks
- neural networks
- social networks

Graph Theory

- Graph theory was founded in 1736 by Leonhard Euler's study of the Seven Bridges of Königsberg problem.
- It remains one of the main areas of discrete mathematics to this day.

More on Euler and the Seven Bridges of Königsberg:



- The Seven Bridges of Königsberg – Numberphile. `https://youtu.be/W18FDEA1jRQ`

Graphs and Directed Graphs – Definitions

Definition (Graph)

A graph (also: undirected graph) is a pair $G = \langle V, E \rangle$, where

- V is a finite set called the set of vertices, and
- $E \subseteq \{\{u, v\} \subseteq V \mid u \neq v\}$ is called the set of edges.

German: Graph, ungerichteter Graph, Knoten, Kanten

Definition (Directed Graph)

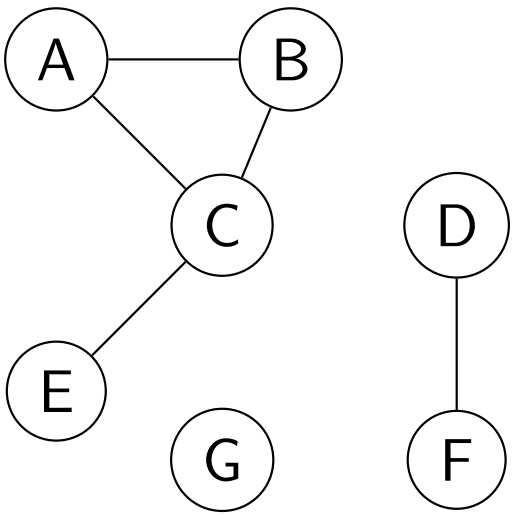
A directed graph (also: digraph) is a pair $G = \langle N, A \rangle$, where

- N is a finite set called the set of nodes, and
- $A \subseteq N \times N$ is called the set of arcs.

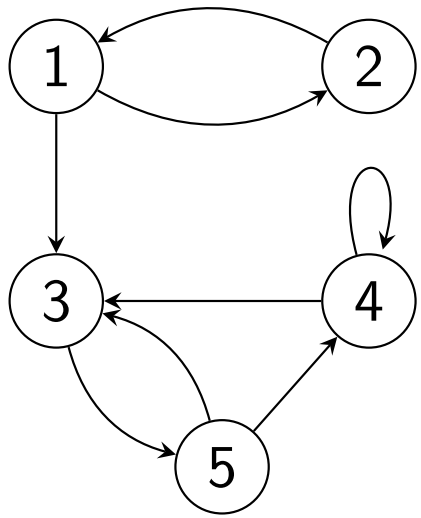
German: gerichteter Graph, Digraph, Knoten, Kanten/Pfeile

Graphs and Directed Graphs – Pictorially

often described pictorially:



graph $\langle V, E \rangle$



directed graph $\langle N, A \rangle$

- $V = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}\}$
- $E = \{\{\mathsf{A}, \mathsf{B}\}, \{\mathsf{A}, \mathsf{C}\}, \{\mathsf{B}, \mathsf{C}\}, \{\mathsf{C}, \mathsf{E}\}, \{\mathsf{D}, \mathsf{F}\}\}$

- $N = \{1, 2, 3, 4, 5\}$
- $A = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 1\rangle, \langle 3, 5\rangle, \langle 4, 3\rangle, \langle 4, 4\rangle, \langle 5, 3\rangle, \langle 5, 4\rangle\}$

Relationship to Relations

graphs vs. directed graphs:

- edges are sets of two elements, arcs are pairs
- arcs can be self-loops $\langle v, v \rangle$; edges cannot (why not?)

(di-)graphs vs. relations:

- A directed graph $\langle N, A \rangle$ is essentially identical to
  (= contains the same information as)
  an arbitrary relation R_A over the finite set N:
  $u \, R_A \, v$ iff $\langle u, v \rangle \in A$
- A graph $\langle V, E \rangle$ is essentially identical to
  an irreflexive symmetric relation R_E over the finite set V:
  $u \, R_E \, v$ iff $\{u, v\} \in E$

Other Kinds of Graphs

many variations exist, for example:

- self-loops may be allowed in edges ("non-simple" graphs)
- labeled graphs: additional information associated with vertices and/or edges
- weighted graphs: numbers associated with edges
- multigraphs: multiple edges between same vertices allowed
- mixed graphs: both edges and arcs allowed
- hypergraphs: edges can involve more than 2 vertices
- infinite graphs: may have infinitely many vertices/edges

Graph Terminology

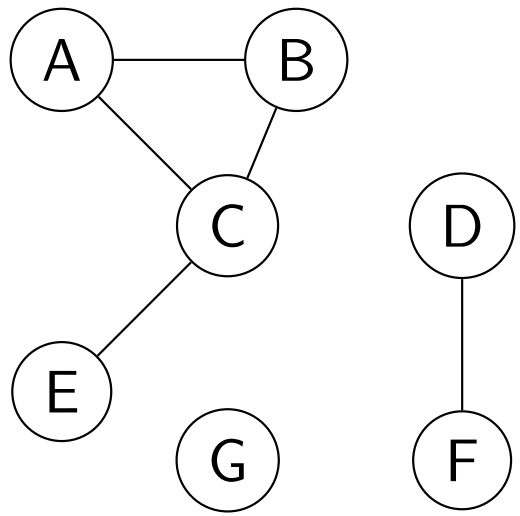
Definition (Graph Terminology)

Let $\langle V, E \rangle$ be a graph.

- u and v are the endpoints of the edge $\{u, v\} \in E$
- u and v are incident to the edge $\{u, v\} \in E$
- u and v are adjacent if $\{u, v\} \in E$
- the vertices adjacent with $v \in V$ are its neighbours $\text{neigh}(v)$: $\text{neigh}(v) = \{w \in V \mid \{v, w\} \in E\}$
- the number of neighbours of $v \in V$ is its degree $\deg(v)$: $\deg(v) = |\text{neigh}(v)|$

German: Endknoten, inzident, adjazent/benachbart, Nachbarn, Grad

Graph Terminology – Examples



endpoints, incident, adjacent, neighbours, degree

Directed Graph Terminology

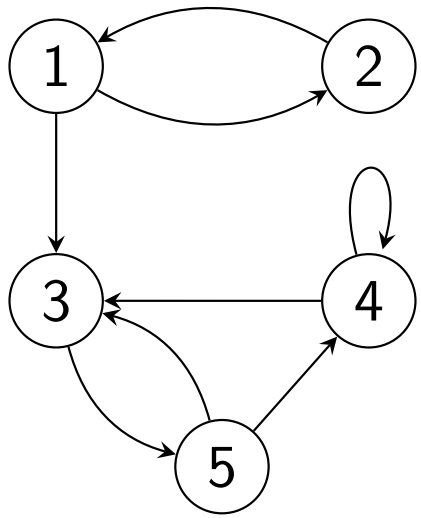
Definition (Directed Graph Terminology)

Let $\langle N, A \rangle$ be a directed graph.

- u is the tail and v is the head of the arc $\langle u, v \rangle \in A$; we say $\langle u, v \rangle$ is an arc from u to v
- u and v are incident to the arc $\langle u, v \rangle \in A$
- u is a predecessor of v and v is a successor of u if $\langle u, v \rangle \in A$
- the predecessors and successor of v are written as $\mathrm{pred}(v) = \{u \in N \mid \langle u, v \rangle \in A\}$ and $\mathrm{succ}(v) = \{w \in N \mid \langle v, w \rangle \in A\}$
- the number of predecessors/successors of $v \in N$ is its indegree/outdegree: $\mathrm{indeg}(v) = |\mathrm{pred}(v)|$, $\mathrm{outdeg}(v) = |\mathrm{succ}(v)|$

German: Fuss, Kopf, inzident, Vorgänger, Nachfolger, Eingangs-/Ausgangsgrad

Directed Graph Terminology – Examples



head, tail, predecessors, successors, indegree, outdegree

C1.2 Induced Graphs and Degree Lemma

Induced Graph of a Directed Graph

Definition (undirected graph induced by a directed graph)

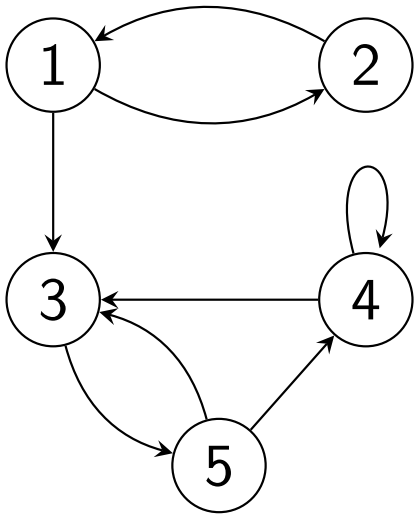
Let $G = \langle N, A \rangle$ be a directed graph.
The (undirected) graph induced by G is the graph $\langle N, E \rangle$ with $E = \{\{u, v\} \mid \langle u, v \rangle \in A, u \neq v\}$.

German: induziert

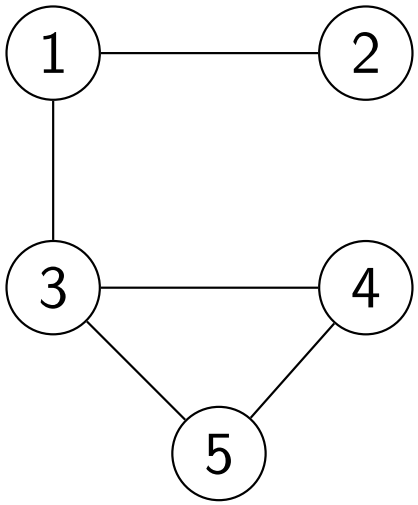
Questions:

- Why require $u \neq v$?
- If $|N| = n$ and $|A| = m$, how many vertices and edges does the induced graph have?
- How does the answer change if G has no self-loops?

Induced Graph of a Directed Graph – Example



- $N = \{1, 2, 3, 4, 5\}$
- $A = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 2, 1\rangle, \langle 3, 5\rangle, \langle 4, 3\rangle, \langle 4, 4\rangle, \langle 5, 3\rangle, \langle 5, 4\rangle\}$



- $V = \{1, 2, 3, 4, 5\}$
- $E = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$

Degree Lemma

Lemma (degree lemma for directed graphs)

Let $\langle N, A \rangle$ be a directed graph.
Then $\sum_{v \in N} \text{indeg}(v) = \sum_{v \in N} \text{outdeg}(v) = |A|$.

Intuitively: every arc contributes 1 to the indegree of one node and 1 to the outdegree of one node.

Lemma (degree lemma for undirected graphs)

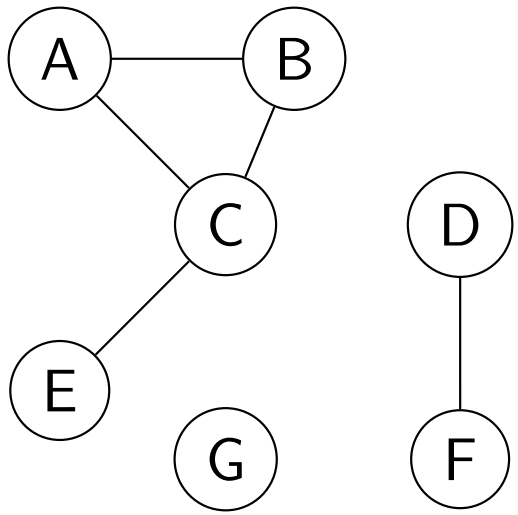
Let $\langle V, E \rangle$ be a graph.
Then $\sum_{v \in V} \deg(v) = 2|E|$.

Intuitively: every edge contributes 1 to the degree of two vertices.

Corollary

Every graph has an even number of vertices with odd degree.

Degree Lemma – Example



$$\sum_{v \in V} \deg(v)$$

$$= \deg(A) + \deg(B) + \deg(C) + \deg(D) + \deg(E) + \deg(F) + \deg(G)$$

$$= 2 + 2 + 3 + 1 + 1 + 1 + 0$$

$$= 10 = 2 \cdot 5 = 2|E|$$

4 vertices with odd degree

Degree Lemma – Proof (1)

Proof of degree lemma for directed graphs.

$$\begin{aligned}
\sum_{v \in N} \mathsf{indeg}(v) &= \sum_{v \in N} |\mathsf{pred}(v)| \\
&= \sum_{v \in N} |\{u \mid u \in N, \langle u, v \rangle \in A\}| \\
&= \sum_{v \in N} |\{\langle u, v \rangle \mid u \in N, \langle u, v \rangle \in A\}| \\
&= \left| \bigcup_{v \in N} \{\langle u, v \rangle \mid u \in N, \langle u, v \rangle \in A\} \right| \\
&= |\{\langle u, v \rangle \mid u \in N, v \in N, \langle u, v \rangle \in A\}| \\
&= |A|.
\end{aligned}$$

$\sum_{v \in N} \mathsf{outdeg}(v) = |A|$ is analogous. □

Degree Lemma – Proof (2)

We omit the proof for undirected graphs,
which can be conducted similarly.

One possible proof strategy that reuses the result we proved:

- Define directed graph $\langle V, A \rangle$ from the graph $\langle V, E \rangle$ by orienting each edge into an arc arbitrarily.
- Observe $\deg(v) = \text{indeg}(v) + \text{outdeg}(v)$, where deg refers to the graph and indeg/outdeg to the directed graph.
- Use the degree lemma for directed graphs:
  $\sum_{v \in V} \deg(v) = \sum_{v \in V} (\text{indeg}(v) + \text{outdeg}(v)) =$
  $\sum_{v \in V} \text{indeg}(v) + \sum_{v \in V} \text{outdeg}(v) = |A| + |A| = 2|A| = 2|E|$