Discrete Structures

B8. Sets: Cardinality and Hilbert's Hotel

Malte Helmert, Gabriele Röger

University of Basel

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$\rightarrow$ How do we compare the cardinality of infinite sets?

Let's first revisit finite sets...

Cardinality of Finite Sets

Cardinality of Sets

A set is finite if it has a finite number of elements.

Definition (Cardinality)

The cardinality of a finite set is the number of elements it contains.

German: Kardinalität oder Mächtigkeit

Cardinality of Sets

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- $|\emptyset| =$
- $|\{x \mid x \in \mathbb{N}_0 \text{ and } 2 \leq x < 5\}| =$
- $|\{3, 0, \{1, 3\}\}| =$
- $|\mathcal{P}(\{1, 2\})| =$

German: Kardinalität oder Mächtigkeit

Cardinality of Power Set

Theorem

Let S be a finite set. Then $|\mathcal{P}(S)| = 2^{|S|}$.

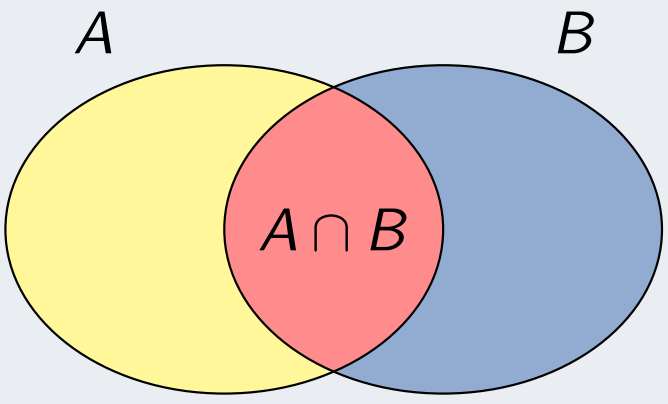
Proof by induction in Ch. A5

Cardinality of the Union of Sets

Theorem

For finite sets A and B it holds that $|A \cup B| = |A| + |B| - |A \cap B|$.

Proof sketch.



$|A| + |B|$ counts $|A \cap B|$ twice.

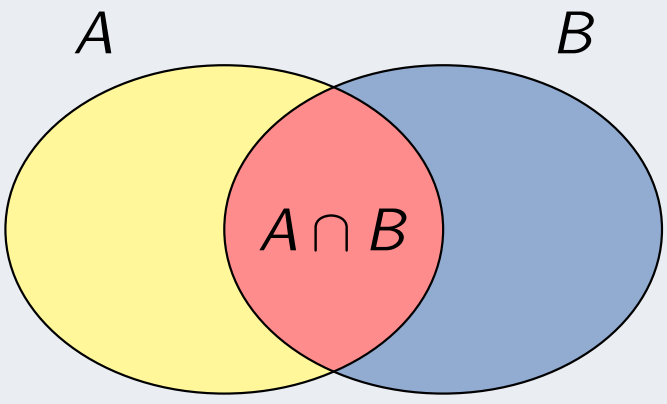
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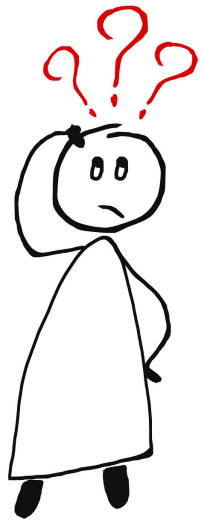
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Corollary

If finite sets A and B are disjoint then $|A \cup B| = |A| + |B|$.

Questions



Questions?

Comparing Cardinality

Comparing the Cardinality of Sets

- Consider $A = \{1, 2\}$ and $B = \{\text{dog}, \text{cat}, \text{mouse}\}$.
- We can map distinct elements of A to distinct elements of B, e.g.

$$1 \mapsto \text{dog}$$
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  - different elements of A are mapped to different elements of B.
- Idea: criterion for B being at least as "large" as A.

Comparing Cardinality

Definition (cardinality not larger)

Set A has cardinality less than or equal to the cardinality of set B ($|A| \leq |B|$), if there is an injective function from A to B.

Comparing the Cardinality of Sets

- $A = \{1, 2, 3\}$ and $B = \{\text{dog}, \text{cat}, \text{mouse}\}$ have cardinality 3.
- We can pair their elements by a bijection from A to B:

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  - Each element of B is paired with exactly one element of A.
- If there is a bijection from A to B there is one from B to A (the inverse function).

Equinumerous Sets

We use the existence of a bijection also as criterion for infinite sets:

Definition (equinumerous sets)

Two sets A and B have the same cardinality ($|A| = |B|$)
if there exists a bijection from A to B.

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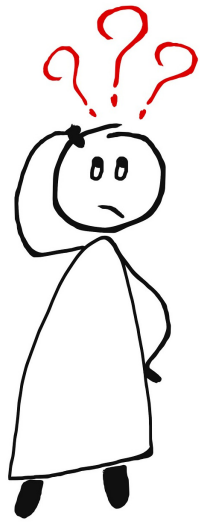
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Consider set A and object $e \notin A$. Is $|A| < |A \cup \{e\}|$?

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Hilbert's Hotel

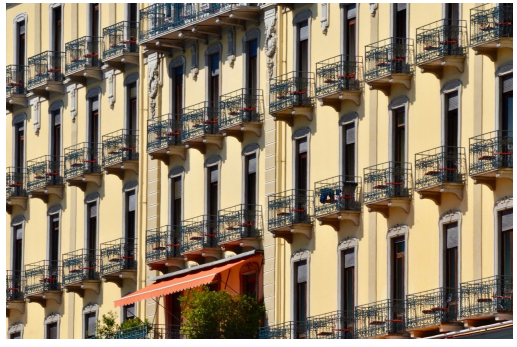
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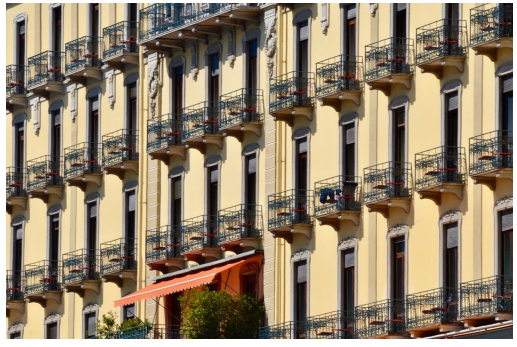
- If in a hotel all rooms are occupied then it cannot accommodate additional guests.



Hilbert's Hotel

Our intuition for finite sets does not always work for infinite sets.

- If in a hotel all rooms are occupied then it cannot accommodate additional guests.
- But Hilbert's Grand Hotel has infinitely many rooms.
- All these rooms are occupied.

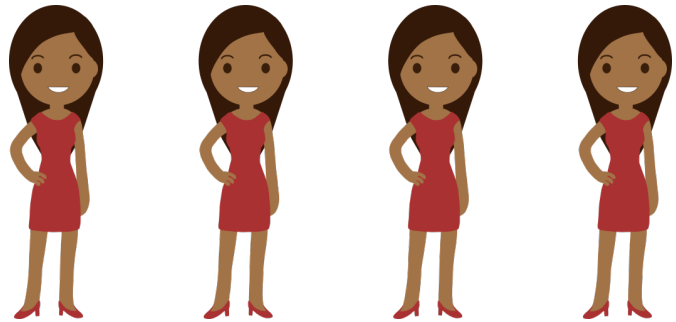


One More Guest Arrives



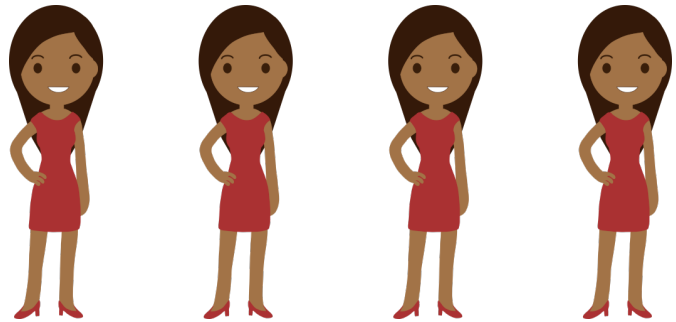
- Every guest moves from her current room n to room $n + 1$.
- Room 1 is then free.
- The new guest gets room 1.

Four More Guests Arrive



- Every guest moves from her current room n to room $n + 4$.
- Rooms 1 to 4 are no longer occupied and can be used for the new guests.

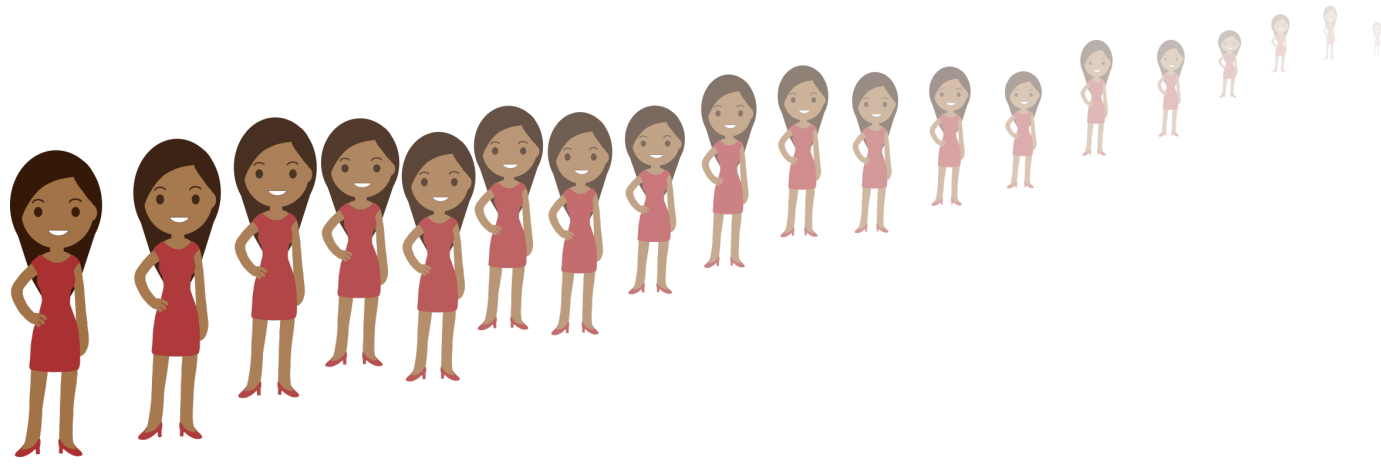
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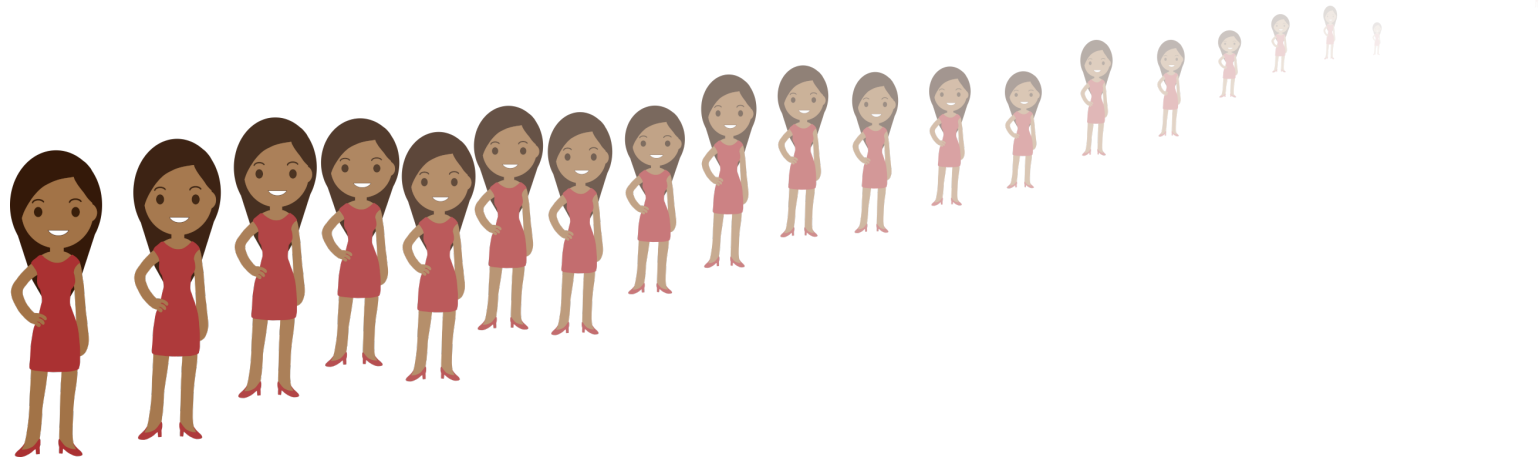
- Every guest moves from her current room n to room $n + 4$.
- Rooms 1 to 4 are no longer occupied and can be used for the new guests.

$\rightarrow$ Works for any finite number of additional guests.

An Infinite Number of Guests Arrives



An Infinite Number of Guests Arrives



- Every guest moves from her current room n to room $2n$.
- The infinitely many rooms with odd numbers are now available.
- The new guests fit into these rooms.

Can we Go further?

What if . . .

- infinitely many coaches, each with an infinite number of guests

. . . arrive?

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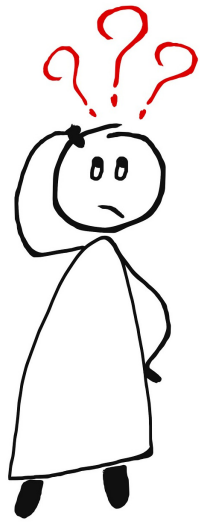
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There are strategies for all these situations
as long as with "infinite" we mean "countably infinite"
and there is a finite number of layers.

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- Set A has cardinality less than or equal to the cardinality of set B ($|A| \leq |B|$), if there is an injective function from A to B.
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- Sets A and B have the same cardinality ($|A| = |B|$) if there exists a bijection from A to B.
- Our intuition for finite sets does not always work for infinite sets.