

Discrete Structures

B5. Operations on Relations

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B5.1 Operations on Relations

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Relations: Recap

- ▶ A **relation over sets** $S_1, \dots, S_n$ is a set $R \subseteq S_1 \times \dots \times S_n$.
- ▶ A **binary** relation is a relation over two sets.
- ▶ A **homogeneous** relation R over set S is a binary relation $R \subseteq S \times S$.

Set Operations

- ▶ Relations are **sets** of tuples, so we can build their union, intersection, complement, ...
- ▶ Let R be a relation over $S_1, \dots, S_n$ and R' a relation over $S'_1, \dots, S'_n$. Then $R \cup R'$ is a relation over $S_1 \cup S'_1, \dots, S_n \cup S'_n$.
With the standard relations $<, =$ and $\leq$ for $\mathbb{N}_0$,
relation $\leq$ corresponds to the union of relations $<$ and $=$.
- ▶ Let R and R' be relations over n sets.
Then $R \cap R'$ is a relation.
Over which sets?
With the standard relations $\leq, =$ and $\geq$ for $\mathbb{N}_0$,
relation $=$ corresponds to the intersection of $\leq$ and $\geq$.
- ▶ If R is a relation over $S_1, \dots, S_n$
then so is the **complementary relation** $\bar{R} = (S_1 \times \dots \times S_n) \setminus R$.
With the standard relations for $\mathbb{N}_0$, relation $=$ is the
complementary relation of $\neq$ and $>$ the one of $\leq$.

Inverse of a Relation

Definition

Let $R \subseteq A \times B$ be a binary relation over A and B .

The **inverse relation** of R is the relation $R^{-1} \subseteq B \times A$ given by
 $R^{-1} = \{(b, a) \mid (a, b) \in R\}$.

- ▶ The inverse of the $<$ relation over $\mathbb{N}_0$ is the $>$ relation.
- ▶ Relation R with xRy iff person x has a key for lock y .
Inverse: Q with aQb iff lock a can be opened by person b .

German: inverse Relation oder Umkehrrelation

Composition of Relations

Definition (Composition of relations)

Let R_1 be a relation over A and B and R_2 a relation over B and C .

The **composition of R_1 and R_2** is the relation $R_2 \circ R_1$ over A and C with:

$$R_2 \circ R_1 = \{(a, c) \mid \text{there is a } b \in B \text{ with} \\ (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

How can we illustrate this graphically?

German: Komposition oder Rückwärtsverkettung

Composition of Relations: Example

$$S_1 = \{1, 2, 3, 4\}$$

$$S_2 = \{A, B, C, D, E\}$$

$$S_3 = \{a, b, c, d\}$$

$$R_1 = \{(1, A), (1, B), (3, B), (4, D)\} \text{ over } S_1 \text{ and } S_2$$

$$R_2 = \{(B, a), (C, c), (D, a), (D, d)\} \text{ over } S_2 \text{ and } S_3$$

$$R_2 \circ R_1 =$$

Composition is Associative

Theorem (Associativity of composition)

Let $S_1, \dots, S_4$ be sets and R_1, R_2, R_3 relations with $R_i \subseteq S_i \times S_{i+1}$.
Then

$$R_3 \circ (R_2 \circ R_1) = (R_3 \circ R_2) \circ R_1.$$

Proof.

It holds that $(x_1, x_4) \in R_3 \circ (R_2 \circ R_1)$ iff there is an x_3 with $(x_1, x_3) \in R_2 \circ R_1$ and $(x_3, x_4) \in R_3$.

As $(x_1, x_3) \in R_2 \circ R_1$ iff there is an x_2 with $(x_1, x_2) \in R_1$ and $(x_2, x_3) \in R_2$, we have overall that $(x_1, x_4) \in R_3 \circ (R_2 \circ R_1)$ iff there are x_2, x_3 with $(x_1, x_2) \in R_1$, $(x_2, x_3) \in R_2$ and $(x_3, x_4) \in R_3$.

This is the case iff there is an x_2 with $(x_1, x_2) \in R_1$ and $(x_2, x_4) \in R_3 \circ R_2$, which holds iff $(x_1, x_4) \in (R_3 \circ R_2) \circ R_1$. $\square$

(Reflexive) Transitive Closure

Definition ((Reflexive) transitive closure)

Let R be a relation over set S .

The **transitive closure** R^+ of R is the **smallest relation over S that is transitive and has R as a subset**.

The **reflexive transitive closure** R^* of R is the **smallest relation over S that is reflexive, transitive and has R as a subset**.

The (reflexive) transitive closure always exists. *Why?*

Example: If aRb specifies that there is a direct flight from a to b , what do R^+ and R^* express?

German: (reflexive) transitive Hülle

Transitive Closure and n -fold Composition

Define the **n -fold composition** of a relation R over S as

$$R_0 = \{(x, x) \mid x \in S\} \quad \text{and} \\ R_i = R \circ R_{i-1} \quad \text{for } i \geq 1.$$

Theorem

Let R be a relation over set S .

Then $R^+ = \bigcup_{i=1}^{\infty} R_i$ and $R^* = \bigcup_{i=0}^{\infty} R_i$.

Without proof.

German: n -fache Komposition

Other Operators

- There are many more operators, also for general relations.
- Highly relevant for **queries over relational databases**.
- For example, **join** operators combine relations based on common entries.
- Example for a **natural join**:

Employee			Dept		Employee ⋈ Dept			
Name	Empld	DeptName	DeptName	Manager	Name	Empld	DeptName	Manager
Harry	3415	Finance	Finance	George	Harry	3415	Finance	George
Sally	2241	Sales	Sales	Harriet	Sally	2241	Sales	Harriet
George	3401	Finance	Production	Charles	George	3401	Finance	George
Harriet	2202	Sales			Harriet	2202	Sales	Harriet
Mary	1257	Human Resources						

(Source: Wikipedia)

Summary

- ▶ Relations: general, binary, homogeneous
- ▶ Properties: reflexivity, symmetry, transitivity
(and related properties)
- ▶ Special relations: equivalence relations, order relations
- ▶ Operations: inverse, composition, transitive closure