

Discrete Structures

B4. Order Relations

Malte Helmert, Gabriele Röger

University of Basel

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Relations: Recap

- A **relation over sets** $S_1, \dots, S_n$ is a set $R \subseteq S_1 \times \dots \times S_n$.
- Possible properties of homogeneous relations R over S :
 - **reflexive**: $\langle x, x \rangle \in R$ for all $x \in S$
 - **irreflexive**: $\langle x, x \rangle \notin R$ for all $x \in S$
 - **symmetric**: $\langle x, y \rangle \in R$ iff $\langle y, x \rangle \in R$
 - **asymmetric**: if $\langle x, y \rangle \in R$ then $\langle y, x \rangle \notin R$
 - **antisymmetric**: if $\langle x, y \rangle \in R$ then $\langle y, x \rangle \notin R$ or $x = y$
 - **transitive**: if $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$ then $\langle x, z \rangle \in R$

Order Relations

- Equivalence relations are reflexive, symmetric and transitive and describe the equivalence of objects (chapter B3).
- We now consider other combinations of properties, that allow us to describe a consistent order of the objects.

German: Ordnungsrelation

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- We now consider other combinations of properties, that allow us to describe a consistent order of the objects.
- “Number x is not larger than number y .”
“Set S is a subset of set T .”
“Jerry runs at least as fast as Tom.”
“Pasta tastes better than potatoes.”

Partial Orders

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- Example partial order relations are $\leq$ over $\mathbb{N}_0$ or $\subseteq$ for sets.
- Are these relations
 - reflexive?
 - irreflexive?
 - symmetric?
 - asymmetric?
 - antisymmetric?
 - transitive?

Partial Orders – Definition

Definition (Partial order, partially ordered set)

A binary relation $\preceq$ over set S is a **partial order** if $\preceq$ is **reflexive, antisymmetric and transitive**.

A **partially ordered set**, or **poset** for short, is a pair $\langle S, \preceq \rangle$ consisting of a set S and a partial order $\preceq$ over S .

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Which of these relations are partial orders?

- strict subset relation $\subset$ for sets
- not-less-than relation $\geq$ over $\mathbb{N}_0$
- $R = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, b \rangle, \langle b, c \rangle, \langle c, c \rangle\}$ over $\{a, b, c\}$

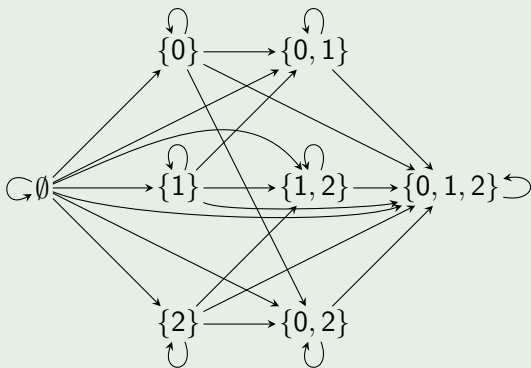
German: Halbordnung oder partielle Ordnung

Graphical Representation of Homogeneous Relations

If S is finite, we can draw any homogeneous relation $R \subseteq S \times S$ as a directed graph:

- one node for each element of S
- arc from node x to node y if $\langle x, y \rangle \in R$

Example ($\subseteq$ over $\mathcal{P}(\{0, 1, 2\})$)

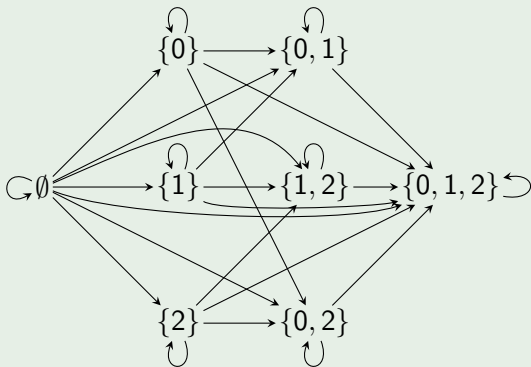


Simplification to a Hasse Diagram (1)

For partial orders, certain arcs always exist.

We can **omit such arcs** from the picture.

Example



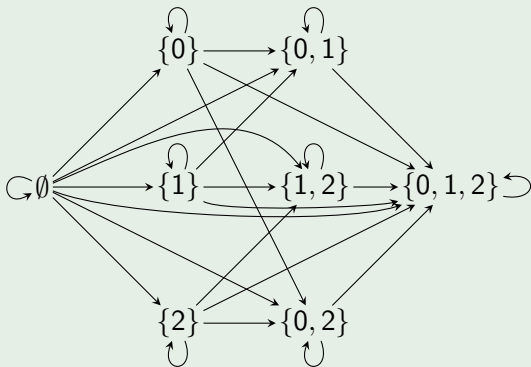
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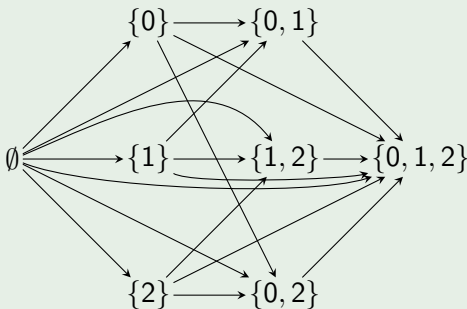
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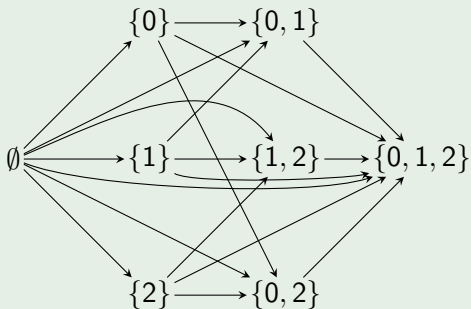
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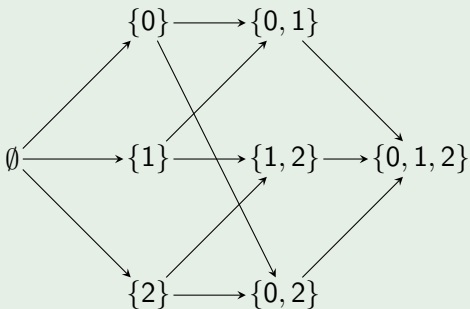
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Simplification to a Hasse Diagram (2)

- **antisymmetric**: if there is an arc from x to y (with $x \neq y$) then there is no arc from y to x .

Omit the arrow head in $x \rightarrow y$ and draw x below y in the diagram.

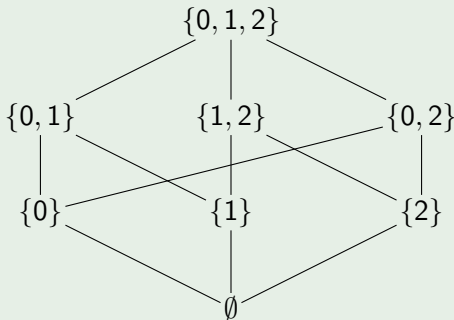
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Hasse Diagram

Compact visual representation of a finite partial order.

Hasse Diagram

Given a partial order relation $\leq$ over a finite set S , a **Hasse diagram** for $\leq$ is a drawing with the elements of S as nodes and with undirected edges such that:

- If there is an edge between x and y , then either x is below y and $x \leq y$, or y is below x and $y \leq x$.
- If $x \leq z$ with $x \neq z$, and no y distinct from both x and z satisfies $x \leq y \leq z$, then there is an edge between x and z .
- There are no other edges.

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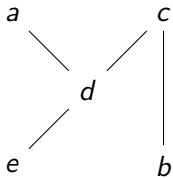
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Reading the diagram

$x \leq y$ iff $x = y$ or there is an upward path from x to y .

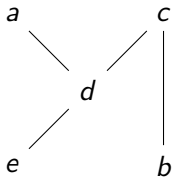
Hasse Diagram: Another Example

Partial order $\leq$ over $\{a, b, c, d, e\}$.



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$$\begin{aligned} \leq = \{ & \langle e, d \rangle, \langle e, a \rangle, \langle e, c \rangle, \langle b, c \rangle, \langle d, a \rangle, \langle d, c \rangle, \\ & \langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle d, d \rangle, \langle e, e \rangle \} \end{aligned}$$

Least and Greatest Element

Definition (Least and greatest element)

Let $\preceq$ be a partial order over set S .

An element $x \in S$ is the **least element** of S if $x \preceq y$ for all $y \in S$.

It is the **greatest element** of S if $y \preceq x$ for all $y \in S$.

German: kleinstes/grösstes Element

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- Is there a least/greatest element? Which one?
 - $S = \{1, 2, 3\}$ and $\preceq = \{(x, y) \mid x, y \in S \text{ and } x \leq y\}$

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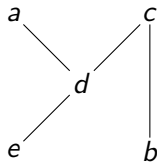
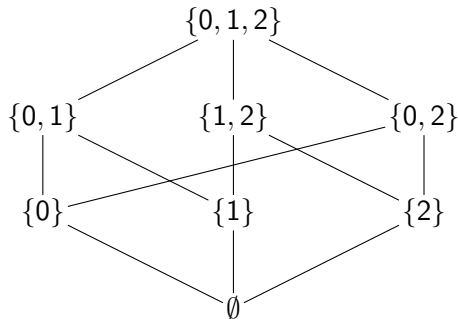
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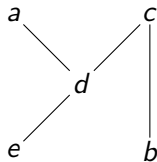
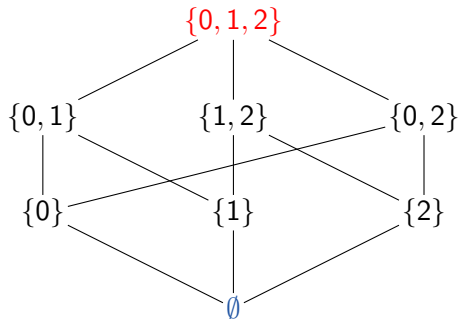
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Are there **least**/**greatest** elements in these partial orders?
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Uniqueness of Least Element

Why can we say **the** least element instead of **a** least element?

Theorem

Let $\preceq$ be a partial order over set S .

If S contains a least element, it contains exactly one least element.

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Analogously: If there is a greatest element then it is unique.

Minimal and Maximal Elements

Definition (Minimal/Maximal element of a set)

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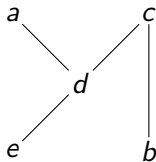
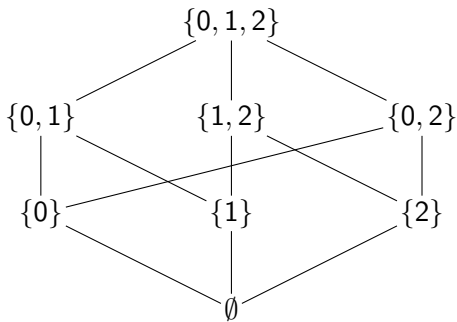
An element $x \in S$ is a **minimal element** of S
if **there is no $y \in S$ with $y \preceq x$ and $x \neq y$.**

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Minimal and Maximal Elements: Example

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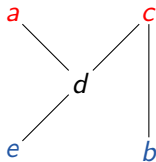
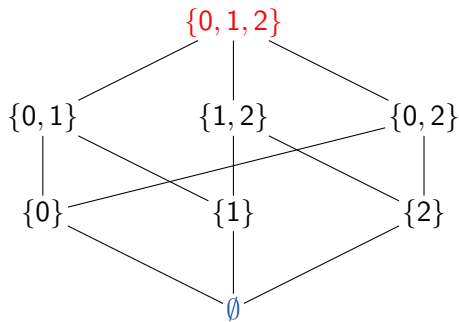
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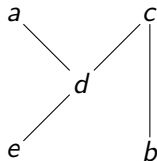
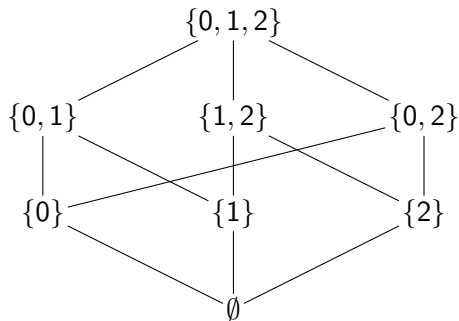
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Which are the **minimal**/**maximal** elements in these partial orders?



A poset can have several minimal elements and no least element.

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 - $\{1, 2\} \not\subseteq \{2, 3\}$ and $\{2, 3\} \not\subseteq \{1, 2\}$
- Relation $\leq$ is a **total** order, relation $\subseteq$ is not.

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Definition (Total order)

A binary relation is a **total order** if it is **total** and a **partial order**.

German: totale Relation, (schwache) Totalordnung oder totale Ordnung

Questions



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We can omit irreflexivity or asymmetry from the definition (but not both). Why?

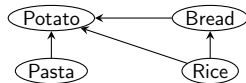
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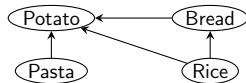
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 - “Pasta tastes better than potato.”
 - “Rice tastes better than bread.”
 - “Bread tastes better than potato.”
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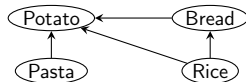


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- **Example 2:** $\subset$ relation for sets

- It **doesn't work** to simply require that the strict order is total.
Why?

Strict Total Orders – Definition

Definition (Trichotomy)

A binary relation R over set S is **trichotomous** if for all $x, y \in S$ exactly one of xRy , yRx or $x = y$ is true.

German: trichotom

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A binary relation $\prec$ over S is a **strict total order** if $\prec$ is **trichotomous** and a **strict order**.

A strict total order completely ranks the elements of set S .

Example: $<$ relation over $\mathbb{N}_0$ gives the standard ordering
 $0, 1, 2, 3, \dots$ of natural numbers.

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Attention: a non-empty strict total order is never a total order.

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Special Elements

Special elements are defined almost as for partial orders:

Definition (Least/greatest/minimal/maximal element)

Let $\prec$ be a **strict** order over set S . An element $x \in S$ is ...

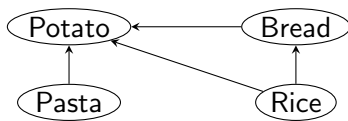
- the **least element** of S if $x \prec y$ for all $y \in S$ **with** $y \neq x$.
- the **greatest element** of S if $y \prec x$ for all $y \in S$ **with** $y \neq x$.
- a **minimal element** of S if there is no $y \in S$ with $y \prec x$.
- a **maximal element** of S if there is no $y \in S$ with $x \prec y$.

Special Elements – Example

Consider again the previous example:

$S = \{\text{Pasta}, \text{Potato}, \text{Bread}, \text{Rice}\}$

$\prec = \{(\text{Pasta}, \text{Potato}), (\text{Bread}, \text{Potato}),$
 $(\text{Rice}, \text{Potato}), (\text{Rice}, \text{Bread})\}$



Is there a least and a greatest element?

Which elements are maximal or minimal?

Questions



Questions?

Summary

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- A partial order $x \preceq y$ is reflexive, antisymmetric and transitive.
 - A Hasse diagram is a compact visual representation of a finite partial order.
 - If x is the greatest element of a set S , it is greater than every element: for all $y \in S$ it holds that $y \preceq x$.
 - If x is a maximal element of set S then it is not smaller than any other element y : there is no $y \in S$ with $x \preceq y$ and $x \neq y$.
 - A total order is a partial order without incomparable objects.

Summary

- A partial order $x \preceq y$ is reflexive, antisymmetric and transitive.
 - A Hasse diagram is a compact visual representation of a finite partial order.
 - If x is the greatest element of a set S , it is greater than every element: for all $y \in S$ it holds that $y \preceq x$.
 - If x is a maximal element of set S then it is not smaller than any other element y : there is no $y \in S$ with $x \preceq y$ and $x \neq y$.
 - A total order is a partial order without incomparable objects.
- A strict order is irreflexive, asymmetric and transitive.
 - Strict total orders and special elements are analogously defined as for partial orders but with a special treatment of equal elements.