

Discrete Structures

B4. Order Relations

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B4.1 Partial Orders

B4.2 Strict Orders

Relations: Recap

- ▶ A **relation over sets** $S_1, \dots, S_n$ is a set $R \subseteq S_1 \times \dots \times S_n$.
- ▶ Possible properties of homogeneous relations R over S :
 - ▶ **reflexive**: $\langle x, x \rangle \in R$ for all $x \in S$
 - ▶ **irreflexive**: $\langle x, x \rangle \notin R$ for all $x \in S$
 - ▶ **symmetric**: $\langle x, y \rangle \in R$ iff $\langle y, x \rangle \in R$
 - ▶ **asymmetric**: if $\langle x, y \rangle \in R$ then $\langle y, x \rangle \notin R$
 - ▶ **antisymmetric**: if $\langle x, y \rangle \in R$ then $\langle y, x \rangle \notin R$ or $x = y$
 - ▶ **transitive**: if $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in R$ then $\langle x, z \rangle \in R$

Order Relations

- ▶ **Equivalence relations** are **reflexive, symmetric and transitive** and describe the equivalence of objects (chapter B3).
- ▶ We now consider **other combinations of properties**, that allow us to describe a **consistent order** of the objects.
- ▶ “Number x is not larger than number y .”
“Set S is a subset of set T .”
“Jerry runs at least as fast as Tom.”
“Pasta tastes better than potatoes.”

German: Ordnungsrelation

B4.1 Partial Orders

Partial Orders

- ▶ We begin with **partial orders**.
- ▶ Example partial order relations are $\leq$ over $\mathbb{N}_0$ or $\subseteq$ for sets.
- ▶ Are these relations
 - ▶ reflexive?
 - ▶ irreflexive?
 - ▶ symmetric?
 - ▶ asymmetric?
 - ▶ antisymmetric?
 - ▶ transitive?

Partial Orders – Definition

Definition (Partial order, partially ordered set)

A binary relation $\preceq$ over set S is a **partial order** if $\preceq$ is **reflexive, antisymmetric and transitive**.

A **partially ordered set**, or **poset** for short, is a pair $\langle S, \preceq \rangle$ consisting of a set S and a partial order $\preceq$ over S .

Which of these relations are partial orders?

- ▶ strict subset relation $\subset$ for sets
- ▶ not-less-than relation $\geq$ over $\mathbb{N}_0$
- ▶ $R = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, b \rangle, \langle b, c \rangle, \langle c, c \rangle\}$ over $\{a, b, c\}$

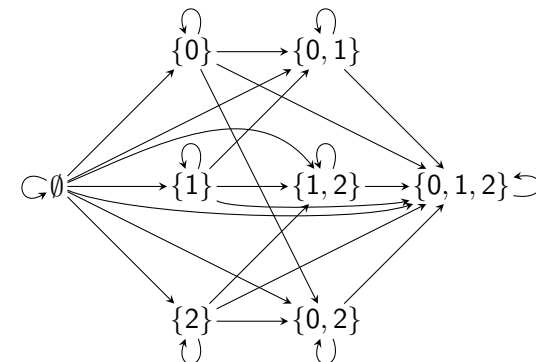
German: Halbordnung oder partielle Ordnung

Graphical Representation of Homogeneous Relations

If S is finite, we can draw any homogeneous relation $R \subseteq S \times S$ as a directed graph:

- ▶ one node for each element of S
- ▶ arc from node x to node y if $\langle x, y \rangle \in R$

Example ($\subseteq$ over $\mathcal{P}(\{0, 1, 2\})$)



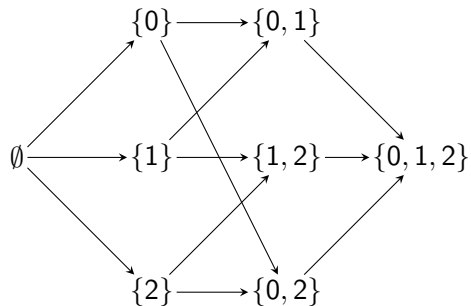
Simplification to a Hasse Diagram (1)

For partial orders, certain arcs always exist.

- **reflexive**: every node has a **self-loop**.
- **transitive**: if there is an arc from x to y and one from y to z then there must be an **arc from x to z** .

We can **omit such arcs** from the picture.

Example

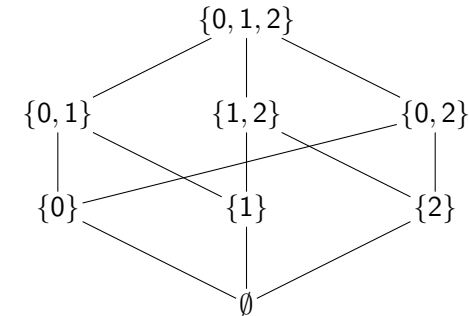


Simplification to a Hasse Diagram (2)

- **antisymmetric**: if there is an arc from x to y (with $x \neq y$) then there is no arc from y to x .

Omit the arrow head in $x \rightarrow y$ and draw x below y in the diagram.

Example



Hasse Diagram

Compact visual representation of a finite partial order.

Hasse Diagram

Given a partial order relation $\leq$ over a finite set S , a **Hasse diagram** for $\leq$ is a drawing with the elements of S as nodes and with undirected edges such that:

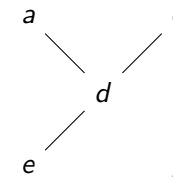
- If there is an edge between x and y , then either x is below y and $x \leq y$, or y is below x and $y \leq x$.
- If $x \leq z$ with $x \neq z$, and no y distinct from both x and z satisfies $x \leq y \leq z$, then there is an edge between x and z .
- There are no other edges.

Reading the diagram

$x \leq y$ iff $x = y$ or there is an upward path from x to y .

Hasse Diagram: Another Example

Partial order $\leq$ over $\{a, b, c, d, e\}$.



$$\leq = \{ \langle e, d \rangle, \langle e, a \rangle, \langle e, c \rangle, \langle b, c \rangle, \langle d, a \rangle, \langle d, c \rangle, \langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle d, d \rangle, \langle e, e \rangle \}$$

Least and Greatest Element

Definition (Least and greatest element)

Let $\preceq$ be a partial order over set S .

An element $x \in S$ is the **least element** of S if $x \preceq y$ for all $y \in S$.

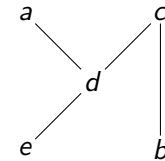
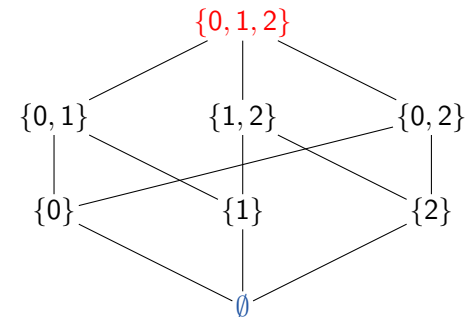
It is the **greatest element** of S if $y \preceq x$ for all $y \in S$.

- Is there a least/greatest element? Which one?
 - $S = \{1, 2, 3\}$ and $\preceq = \{\langle x, y \rangle \mid x, y \in S \text{ and } x \leq y\}$
 - relation $\leq$ over $\mathbb{N}_0$
 - relation $\leq$ over $\mathbb{Z}$

German: kleinstes/grösstes Element

Least and Greatest Element

Are there **least/greatest** elements in these partial orders?
Which ones?



Uniqueness of Least Element

Why can we say **the** least element instead of **a** least element?

Theorem

Let $\preceq$ be a partial order over set S .

If S contains a least element, it contains exactly one least element.

Proof.

By contradiction: Assume x, y are least elements of S with $x \neq y$.

Since x is a least element, $x \preceq y$ is true.

Since y is a least element, $y \preceq x$ is true.

As a partial order is antisymmetric, this implies that $x = y$. $\nmid$ $\square$

Analogously: If there is a greatest element then it is unique.

Minimal and Maximal Elements

Definition (Minimal/Maximal element of a set)

Let $\preceq$ be a partial order over set S .

An element $x \in S$ is a **minimal element** of S

if **there is no** $y \in S$ with $y \preceq x$ and $x \neq y$.

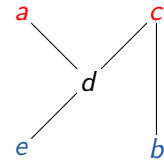
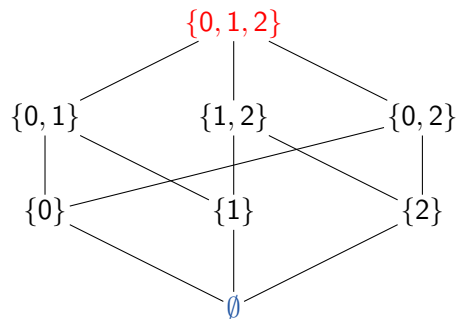
An element $x \in S$ is a **maximal element** of S

if **there is no** $y \in S$ with $x \preceq y$ and $x \neq y$.

German: minimales/maximales Element

Minimal and Maximal Elements: Example

Which are the **minimal**/**maximal** elements in these partial orders?



A poset can have several minimal elements and no least element.

Reminder:

$x \in S$ is a minimal element if there is no $y \in S$ with $y \preceq x$ and $x \neq y$.

$x \in S$ is a maximal element if there is no $y \in S$ with $x \preceq y$ and $x \neq y$.

Total Orders

- ▶ Relations $\leq$ over $\mathbb{N}_0$ and $\subseteq$ for sets are partial orders.
- ▶ Can we compare every object against every object?
 - ▶ For all $x, y \in \mathbb{N}_0$ it holds that $x \leq y$ or that $y \leq x$ (or both).
 - ▶ $\{1, 2\} \not\subseteq \{2, 3\}$ and $\{2, 3\} \not\subseteq \{1, 2\}$
- ▶ Relation $\leq$ is a **total** order, relation $\subseteq$ is not.

Total Order – Definition

Definition (Total relation)

A binary relation R over set S is **total** if for all $x, y \in S$ at least one of xRy or yRx is true.

Definition (Total order)

A binary relation is a **total order** if it is **total** and a **partial order**.

German: totale Relation, (schwache) Totalordnung oder totale Ordnung

B4.2 Strict Orders

Strict Orders

- ▶ A **partial** order is reflexive, antisymmetric and transitive.
- ▶ We now consider **strict orders**.
- ▶ Example strict order relations are $<$ over $\mathbb{N}_0$ or $\subset$ for sets.
- ▶ Are these relations
 - ▶ reflexive?
 - ▶ irreflexive?
 - ▶ symmetric?
 - ▶ asymmetric?
 - ▶ antisymmetric?
 - ▶ transitive?

Strict Orders – Definition

Definition (Strict (partial) order)

A binary relation $\prec$ over set S is a **strict (partial) order** if $\prec$ is **irreflexive, asymmetric and transitive**.

Which of these relations are strict orders?

- ▶ subset relation $\subseteq$ for sets
- ▶ strict superset relation $\supset$ for sets

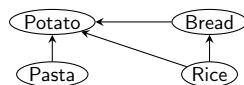
Can a relation be both, a partial order and a strict (partial) order?

We can omit irreflexivity or asymmetry from the definition (but not both). Why?

German: strenge (Halb-)ordnung

Strict Total Orders

- ▶ As with partial orders, a strict order does not necessarily allow us to compare any two objects.
- ▶ **Example 1** (personal preferences):
 - ▶ “Pasta tastes better than potato.”
 - ▶ “Rice tastes better than bread.”
 - ▶ “Bread tastes better than potato.”
 - ▶ “Rice tastes better than potato.”
 - ▶ This definition of “tastes better than” is a strict order.
 - ▶ No ranking of pasta against rice or bread.
- ▶ **Example 2:** $\subset$ relation for sets
- ▶ It **doesn't work** to simply require that the strict order is total. Why?



Strict Total Orders – Definition

Definition (Trichotomy)

A binary relation R over set S is **trichotomous** if for all $x, y \in S$ exactly one of xRy , yRx or $x = y$ is true.

Definition (Strict total order)

A binary relation $\prec$ over S is a **strict total order** if $\prec$ is **trichotomous** and a **strict order**.

A strict total order completely ranks the elements of set S .

Example: $<$ relation over $\mathbb{N}_0$ gives the standard ordering $0, 1, 2, 3, \dots$ of natural numbers.

Attention: a non-empty strict total order is never a total order.

German: trichotom, strenge Totalordnung

Special Elements

Special elements are defined almost as for partial orders:

Definition (Least/greatest/minimal/maximal element)

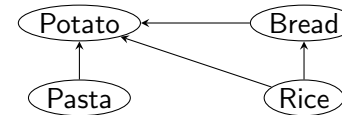
Let $\prec$ be a **strict** order over set S . An element $x \in S$ is ...

- ▶ the **least element** of S if $x \prec y$ for all $y \in S$ with $y \neq x$.
- ▶ the **greatest element** of S if $y \prec x$ for all $y \in S$ with $y \neq x$.
- ▶ a **minimal element** of S if there is no $y \in S$ with $y \prec x$.
- ▶ a **maximal element** of S if there is no $y \in S$ with $x \prec y$.

Special Elements – Example

Consider again the previous example:

$S = \{\text{Pasta, Potato, Bread, Rice}\}$
 $\prec = \{(\text{Pasta, Potato}), (\text{Bread, Potato}),$
 $(\text{Rice, Potato}), (\text{Rice, Bread})\}$



Is there a least and a greatest element?
 Which elements are maximal or minimal?

Summary

- ▶ A **partial order** $x \preceq y$ is **reflexive, antisymmetric and transitive**.
 - ▶ A **Hasse diagram** is a compact visual representation of a finite partial order.
 - ▶ If x is the **greatest element** of a set S , it is greater than every element: for all $y \in S$ it holds that $y \preceq x$.
 - ▶ If x is a **maximal element** of set S then it is not smaller than any other element y : there is no $y \in S$ with $x \preceq y$ and $x \neq y$.
 - ▶ A **total order** is a partial order without incomparable objects.
- ▶ A **strict order** is **irreflexive, asymmetric and transitive**.
 - ▶ Strict **total orders** and **special elements** are analogously defined as for partial orders but with a special treatment of equal elements.