

# Discrete Structures

## B4. Order Relations

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## B4.1 Partial Orders

## B4.2 Strict Orders

# Relations: Recap

- ▶ A **relation over sets**  $S_1, \dots, S_n$  is a set  $R \subseteq S_1 \times \dots \times S_n$ .
- ▶ Possible properties of homogeneous relations  $R$  over  $S$ :
  - ▶ **reflexive**:  $\langle x, x \rangle \in R$  for all  $x \in S$
  - ▶ **irreflexive**:  $\langle x, x \rangle \notin R$  for all  $x \in S$
  - ▶ **symmetric**:  $\langle x, y \rangle \in R$  iff  $\langle y, x \rangle \in R$
  - ▶ **asymmetric**: if  $\langle x, y \rangle \in R$  then  $\langle y, x \rangle \notin R$
  - ▶ **antisymmetric**: if  $\langle x, y \rangle \in R$  then  $\langle y, x \rangle \notin R$  or  $x = y$
  - ▶ **transitive**: if  $\langle x, y \rangle \in R$  and  $\langle y, z \rangle \in R$  then  $\langle x, z \rangle \in R$

# Order Relations

- ▶ Equivalence relations are reflexive, symmetric and transitive and describe the equivalence of objects (chapter B3).
- ▶ We now consider other combinations of properties, that allow us to describe a consistent order of the objects.
- ▶ “Number  $x$  is not larger than number  $y$ .”  
“Set  $S$  is a subset of set  $T$ .”  
“Jerry runs at least as fast as Tom.”  
“Pasta tastes better than potatoes.”

German: Ordnungsrelation

# B4.1 Partial Orders

# Partial Orders

- ▶ We begin with **partial orders**.
- ▶ Example partial order relations are  $\leq$  over  $\mathbb{N}_0$  or  $\subseteq$  for sets.
- ▶ Are these relations
  - ▶ reflexive?
  - ▶ irreflexive?
  - ▶ symmetric?
  - ▶ asymmetric?
  - ▶ antisymmetric?
  - ▶ transitive?

# Partial Orders – Definition

## Definition (Partial order, partially ordered set)

A binary relation  $\preceq$  over set  $S$  is a **partial order** if  $\preceq$  is **reflexive, antisymmetric and transitive**.

A **partially ordered set**, or **poset** for short, is a pair  $\langle S, \preceq \rangle$  consisting of a set  $S$  and a partial order  $\preceq$  over  $S$ .

Which of these relations are partial orders?

- ▶ strict subset relation  $\subset$  for sets
- ▶ not-less-than relation  $\geq$  over  $\mathbb{N}_0$
- ▶  $R = \{\langle a, a \rangle, \langle a, b \rangle, \langle b, b \rangle, \langle b, c \rangle, \langle c, c \rangle\}$  over  $\{a, b, c\}$

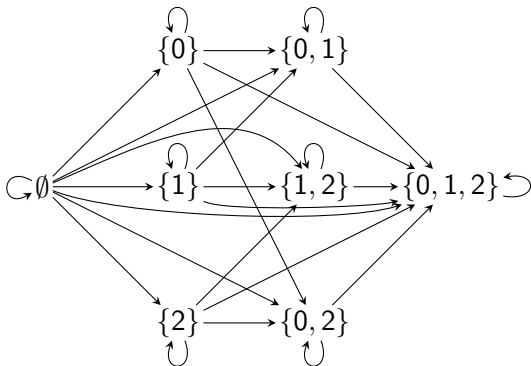
German: Halbordnung oder partielle Ordnung

# Graphical Representation of Homogeneous Relations

If  $S$  is finite, we can draw any homogeneous relation  $R \subseteq S \times S$  as a directed graph:

- ▶ one node for each element of  $S$
- ▶ arc from node  $x$  to node  $y$  if  $\langle x, y \rangle \in R$

Example ( $\subseteq$  over  $\mathcal{P}(\{0, 1, 2\})$ )



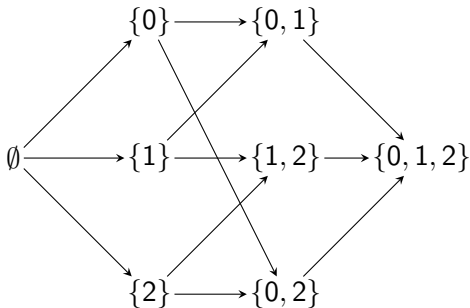
# Simplification to a Hasse Diagram (1)

For partial orders, certain arcs always exist.

- ▶ **reflexive**: every node has a **self-loop**.
- ▶ **transitive**: if there is an arc from  $x$  to  $y$  and one from  $y$  to  $z$  then there must be an **arc from  $x$  to  $z$** .

We can **omit such arcs** from the picture.

## Example

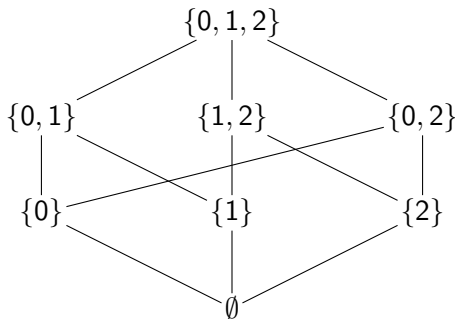


## Simplification to a Hasse Diagram (2)

- **antisymmetric**: if there is an arc from  $x$  to  $y$  (with  $x \neq y$ ) then there is no arc from  $y$  to  $x$ .

Omit the arrow head in  $x \rightarrow y$  and draw  $x$  below  $y$  in the diagram.

### Example



# Hasse Diagram

Compact visual representation of a finite partial order.

## Hasse Diagram

Given a partial order relation  $\leq$  over a finite set  $S$ , a **Hasse diagram** for  $\leq$  is a drawing with the elements of  $S$  as nodes and with undirected edges such that:

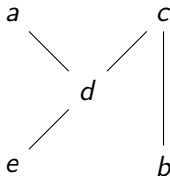
- ▶ If there is an edge between  $x$  and  $y$ , then either  $x$  is below  $y$  and  $x \leq y$ , or  $y$  is below  $x$  and  $y \leq x$ .
- ▶ If  $x \leq z$  with  $x \neq z$ , and no  $y$  distinct from both  $x$  and  $z$  satisfies  $x \leq y \leq z$ , then there is an edge between  $x$  and  $z$ .
- ▶ There are no other edges.

## Reading the diagram

$x \leq y$  iff  $x = y$  or there is an upward path from  $x$  to  $y$ .

## Hasse Diagram: Another Example

Partial order  $\leq$  over  $\{a, b, c, d, e\}$ .



$$\begin{aligned} \leq = \{ & \langle e, d \rangle, \langle e, a \rangle, \langle e, c \rangle, \langle b, c \rangle, \langle d, a \rangle, \langle d, c \rangle, \\ & \langle a, a \rangle, \langle b, b \rangle, \langle c, c \rangle, \langle d, d \rangle, \langle e, e \rangle \} \end{aligned}$$

# Least and Greatest Element

## Definition (Least and greatest element)

Let  $\preceq$  be a partial order over set  $S$ .

An element  $x \in S$  is the **least element** of  $S$  if  $x \preceq y$  for all  $y \in S$ .

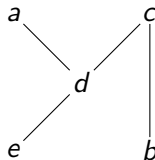
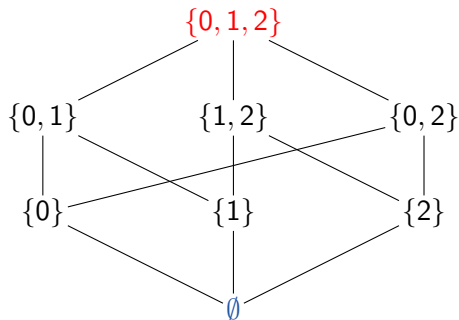
It is the **greatest element** of  $S$  if  $y \preceq x$  for all  $y \in S$ .

- ▶ Is there a least/greatest element? Which one?
  - ▶  $S = \{1, 2, 3\}$  and  $\preceq = \{\langle x, y \rangle \mid x, y \in S \text{ and } x \leq y\}$
  - ▶ relation  $\leq$  over  $\mathbb{N}_0$
  - ▶ relation  $\leq$  over  $\mathbb{Z}$

German: kleinstes/grösstes Element

# Least and Greatest Element

Are there **least**/**greatest** elements in these partial orders?  
Which ones?



# Uniqueness of Least Element

Why can we say **the** least element instead of **a** least element?

## Theorem

*Let  $\preceq$  be a partial order over set  $S$ .*

*If  $S$  contains a least element, it contains exactly one least element.*

## Proof.

**By contradiction:** Assume  $x, y$  are least elements of  $S$  with  $x \neq y$ .

Since  $x$  is a least element,  $x \preceq y$  is true.

Since  $y$  is a least element,  $y \preceq x$  is true.

As a partial order is antisymmetric, this implies that  $x = y$ .  $\nLeftarrow$   $\square$

Analogously: If there is a greatest element then it is unique.

# Minimal and Maximal Elements

## Definition (Minimal/Maximal element of a set)

Let  $\preceq$  be a partial order over set  $S$ .

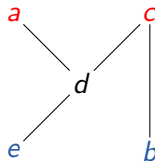
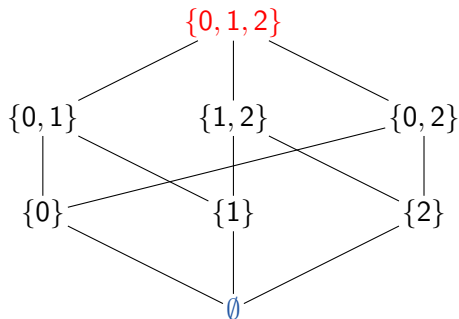
An element  $x \in S$  is a **minimal element** of  $S$   
if **there is no  $y \in S$  with  $y \preceq x$  and  $x \neq y$ .**

An element  $x \in S$  is a **maximal element** of  $S$   
if **there is no  $y \in S$  with  $x \preceq y$  and  $x \neq y$ .**

German: minimales/maximales Element

# Minimal and Maximal Elements: Example

Which are the **minimal**/**maximal** elements in these partial orders?



A poset can have several minimal elements and no least element.

Reminder:

$x \in S$  is a minimal element if there is no  $y \in S$  with  $y \preceq x$  and  $x \neq y$ .

$x \in S$  is a maximal element if there is no  $y \in S$  with  $x \preceq y$  and  $x \neq y$ .

# Total Orders

- ▶ Relations  $\leq$  over  $\mathbb{N}_0$  and  $\subseteq$  for sets are partial orders.
- ▶ Can we compare every object against every object?
  - ▶ For all  $x, y \in \mathbb{N}_0$  it holds that  $x \leq y$  or that  $y \leq x$  (or both).
  - ▶  $\{1, 2\} \not\subseteq \{2, 3\}$  and  $\{2, 3\} \not\subseteq \{1, 2\}$
- ▶ Relation  $\leq$  is a **total** order, relation  $\subseteq$  is not.

# Total Order – Definition

## Definition (Total relation)

A binary relation  $R$  over set  $S$  is **total** if for all  $x, y \in S$  at least one of  $xRy$  or  $yRx$  is true.

## Definition (Total order)

A binary relation is a **total order** if it is **total** and a **partial order**.

German: totale Relation, (schwache) Totalordnung oder totale Ordnung

## B4.2 Strict Orders

# Strict Orders

- ▶ A **partial** order is reflexive, antisymmetric and transitive.
- ▶ We now consider **strict orders**.
- ▶ Example strict order relations are  $<$  over  $\mathbb{N}_0$  or  $\subset$  for sets.
- ▶ Are these relations
  - ▶ reflexive?
  - ▶ irreflexive?
  - ▶ symmetric?
  - ▶ asymmetric?
  - ▶ antisymmetric?
  - ▶ transitive?

# Strict Orders – Definition

## Definition (Strict (partial) order)

A binary relation  $\prec$  over set  $S$  is a **strict (partial) order** if  $\prec$  is **irreflexive, asymmetric and transitive**.

Which of these relations are strict orders?

- ▶ subset relation  $\subseteq$  for sets
- ▶ strict superset relation  $\supset$  for sets

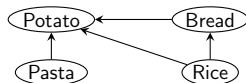
Can a relation be both, a partial order and a strict (partial) order?

We can omit irreflexivity or asymmetry from the definition (but not both). Why?

German: strenge (Halb-)ordnung

# Strict Total Orders

- ▶ As with partial orders, a strict order does not necessarily allow us to compare any two objects.
- ▶ **Example 1** (personal preferences):
  - ▶ “Pasta tastes better than potato.”
  - ▶ “Rice tastes better than bread.”
  - ▶ “Bread tastes better than potato.”
  - ▶ “Rice tastes better than potato.”
  - ▶ This definition of “tastes better than” is a strict order.
  - ▶ No ranking of pasta against rice or bread.
- ▶ **Example 2:**  $\subset$  relation for sets
- ▶ It **doesn't work** to simply require that the strict order is total.  
Why?



# Strict Total Orders – Definition

## Definition (Trichotomy)

A binary relation  $R$  over set  $S$  is **trichotomous** if for all  $x, y \in S$  exactly one of  $xRy$ ,  $yRx$  or  $x = y$  is true.

## Definition (Strict total order)

A binary relation  $\prec$  over  $S$  is a **strict total order** if  $\prec$  is **trichotomous** and a **strict order**.

A strict total order completely ranks the elements of set  $S$ .

**Example:**  $<$  relation over  $\mathbb{N}_0$  gives the standard ordering  
 $0, 1, 2, 3, \dots$  of natural numbers.

**Attention:** a non-empty strict total order is never a total order.

German: trichotom, strenge Totalordnung

# Special Elements

Special elements are defined almost as for partial orders:

## Definition (Least/greatest/minimal/maximal element)

Let  $\prec$  be a **strict** order over set  $S$ . An element  $x \in S$  is ...

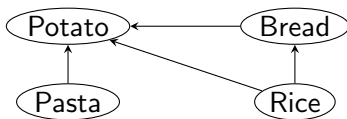
- ▶ the **least element** of  $S$  if  $x \prec y$  for all  $y \in S$  **with**  $y \neq x$ .
- ▶ the **greatest element** of  $S$  if  $y \prec x$  for all  $y \in S$  **with**  $y \neq x$ .
- ▶ a **minimal element** of  $S$  if there is no  $y \in S$  with  $y \prec x$ .
- ▶ a **maximal element** of  $S$  if there is no  $y \in S$  with  $x \prec y$ .

## Special Elements – Example

Consider again the previous example:

$S = \{\text{Pasta}, \text{Potato}, \text{Bread}, \text{Rice}\}$

$\prec = \{(\text{Pasta}, \text{Potato}), (\text{Bread}, \text{Potato}),$   
 $(\text{Rice}, \text{Potato}), (\text{Rice}, \text{Bread})\}$



Is there a least and a greatest element?

Which elements are maximal or minimal?

# Summary

- ▶ A **partial order**  $x \preceq y$  is **reflexive, antisymmetric and transitive**.
  - ▶ A **Hasse diagram** is a compact visual representation of a finite partial order.
  - ▶ If  $x$  is the **greatest element** of a set  $S$ , it is greater than every element: for all  $y \in S$  it holds that  $y \preceq x$ .
  - ▶ If  $x$  is a **maximal element** of set  $S$  then it is not smaller than any other element  $y$ : there is no  $y \in S$  with  $x \preceq y$  and  $x \neq y$ .
  - ▶ A **total order** is a partial order without incomparable objects.
- ▶ A **strict order** is **irreflexive, asymmetric and transitive**.
  - ▶ Strict **total orders** and **special elements** are analogously defined as for partial orders but with a special treatment of equal elements.