

Discrete Structures

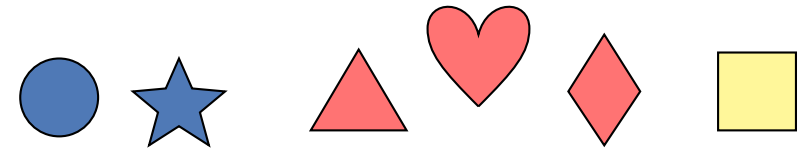
B3. Equivalence Relations and Partitions

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Colored Shapes



Consider the **color** of the objects.

Write $x \sim_C y$ to express that objects x and y have the same color.

circle $\sim_C$ star

star $\sim_C$ circle

circle $\sim_C$ circle

star $\sim_C$ star

triangle $\sim_C$ heart

triangle $\sim_C$ diamond

diamond $\sim_C$ heart

heart $\sim_C$ triangle

diamond $\sim_C$ triangle

heart $\sim_C$ diamond

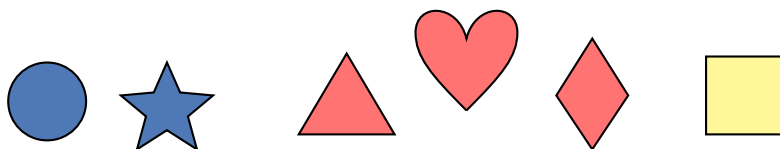
square $\sim_C$ square

heart $\sim_C$ heart

diamond $\sim_C$ diamond

triangle $\sim_C$ triangle

Colored Shapes



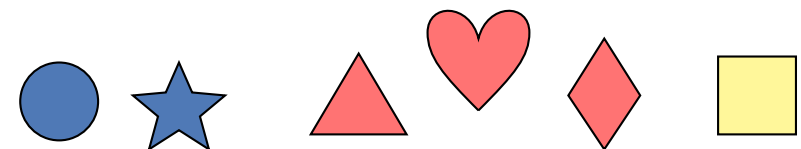
Consider the **color** of the objects.

Write $[x]_C$ for the set of all objects that have the same color as x .

$$[\text{heart}]_C = \{\text{heart}, \text{triangle}, \text{diamond}\} = [\text{triangle}]_C = [\text{diamond}]_C$$

We call this the **equivalence class** of heart.

Colored Shapes



Consider the **shape** of the objects.

Write $x \sim_S y$ to express that objects x and y have the same shape.

circle $\sim_S$ circle

star $\sim_S$ star

triangle $\sim_S$ triangle

heart $\sim_S$ heart

diamond $\sim_S$ diamond

square $\sim_S$ square

Motivation

- ▶ Think of any attribute that two objects can have in common, e. g. their color.
- ▶ We could place the objects into distinct “buckets”, e. g. one bucket for each color.
- ▶ We also can define a relation $\sim$ such that $x \sim y$ iff x and y share the attribute, e. g. have the same color.
- ▶ Would this relation be
 - ▶ reflexive?
 - ▶ irreflexive?
 - ▶ symmetric?
 - ▶ asymmetric?
 - ▶ antisymmetric?
 - ▶ transitive?

Equivalence Relation

Definition (Equivalence Relation)

A binary relation $\sim$ over set S is an **equivalence relation** if $\sim$ is **reflexive, symmetric and transitive**.

Examples:

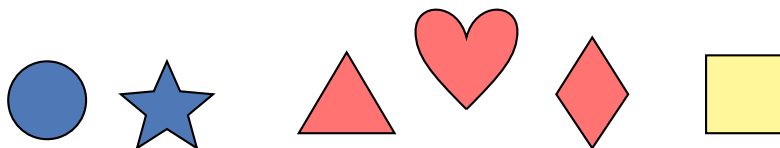
- ▶ $\{(x, y) \mid x \text{ and } y \text{ have the same place of origin}\}$ over the set of all Swiss citizens
- ▶ $\{(x, y) \mid x \text{ and } y \text{ have the same parity}\}$ over $\mathbb{N}_0$
- ▶ $\{(1, 1), (1, 4), (1, 5), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), (2, 2), (2, 3), (3, 2), (3, 3)\}$ over $\{1, 2, \dots, 5\}$

Is this definition indeed what we want?

Does it allow us to partition the objects into buckets (e. g. one “bucket” for all objects that share a specific color)?

German: Äquivalenzrelation

Partition (by Color)



$\{\{\text{circle, star}\}, \{\text{triangle, heart, diamond}\}, \{\text{square}\}\}$

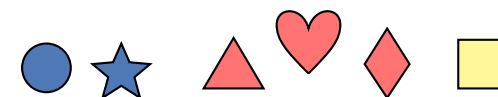
Partition

Definition (Partition)

A **partition** of a set S is a set $P \subseteq \mathcal{P}(S)$ such that

- ▶ $X \neq \emptyset$ for all $X \in P$,
- ▶ $\bigcup_{X \in P} X = S$, and
- ▶ $X \cap Y = \emptyset$ for all $X, Y \in P$ with $X \neq Y$,

The elements of P are called the **blocks** of the partition.



$\{\{\text{circle, star}\}, \{\text{triangle, heart, diamond}\}, \{\text{square}\}\}$

Partition

Let $S = \{e_1, \dots, e_5\}$.

Which of the following sets are partitions of S ?

- ▶ $P_1 = \{\{e_1, e_4\}, \{e_3\}, \{e_2, e_5\}\}$
- ▶ $P_2 = \{\{e_1, e_4, e_5\}, \{e_3\}\}$
- ▶ $P_3 = \{\{e_1, e_4, e_5\}, \{e_3\}, \{e_2, e_5\}\}$
- ▶ $P_4 = \{\{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_5\}\}$
- ▶ $P_5 = \{\{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \{e_5\}, \{\}\}$

A Property of Partitions

Lemma

Let S be a set and P be a partition of S .

Then every $x \in S$ is an element of exactly one $X \in P$.

Proof: $\leadsto$ homework

Block of an Element

The lemma enables the following definition:

Definition

Let S be a set and P be a partition of S .

For $e \in S$ we denote by $[e]_P$ the block $X \in P$ such that $e \in X$.

Consider partition $P = \{\{e_1, e_4\}, \{e_3\}, \{e_2, e_5\}\}$ of $\{e_1, \dots, e_5\}$.

$[e_1]_P =$

Connection between Partitions and Equivalence Relations?

- ▶ We will now explore the connection between partitions and equivalence relations.
- ▶ **Spoiler:** They are essentially the same concept.

Partitions Induce Equivalence Relations I

Definition (Relation induced by a partition)

Let S be a set and P be a partition of S .

The **relation $\sim_P$ induced by P** is the binary relation over S with

$$x \sim_P y \text{ iff } [x]_P = [y]_P.$$

$x \sim_P y$ iff x and y are in the same block of P .

Consider partition $P = \{\{1, 4, 5\}, \{2, 3\}\}$ of set $\{1, 2, \dots, 5\}$.

$$\sim_P = \{(1, 1), (1, 4), (1, 5), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

We will show that $\sim_P$ is an equivalence relation.

Partitions Induce Equivalence Relations II

Theorem

Let P be a partition of S .

Relation $\sim_P$ induced by P is an **equivalence relation** over S .

Proof.

We need to show that $\sim_P$ is reflexive, symmetric and transitive.

reflexive: As $=$ is reflexive it holds for all $x \in S$ that $[x]_P = [x]_P$ and hence also that $x \sim_P x$.

symmetric: If $x \sim_P y$ then $[x]_P = [y]_P$. With the symmetry of $=$ we get that $[y]_P = [x]_P$ and conclude that $y \sim_P x$.

transitive: If $x \sim_P y$ and $y \sim_P z$ then $[x]_P = [y]_P$ and $[y]_P = [z]_P$. As $=$ is transitive, it then also holds that $[x]_P = [z]_P$ and hence $x \sim_P z$. $\square$

Equivalence Classes

Definition (Equivalence Class)

Let $\sim$ be an equivalence relation over set S .

For any $x \in S$, the **equivalence class of x** is the set

$$[x]_{\sim} = \{y \in S \mid x \sim y\}.$$

Any $r \in [x]_{\sim}$ is called a **representative** of $[x]_{\sim}$.

Consider

$$\sim = \{(1, 1), (1, 4), (1, 5), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

over set $\{1, 2, \dots, 5\}$.

$$[4]_{\sim} =$$

German: Äquivalenzklasse

Equivalence Relations Induce Partitions

Theorem

Let R be an equivalence relation over set S .

The set $P = \{[x]_R \mid x \in S\}$ is a **partition of S** .

Proof.

We need to show that

- ① $X \neq \emptyset$ for all $X \in P$,
- ② $\bigcup_{X \in P} X = S$, and
- ③ $X \cap Y = \emptyset$ for all $X, Y \in P$ with $X \neq Y$,

1) For $x \in S$, it holds that $x \in [x]_R$ because R is reflexive.
Hence, no $X \in P$ is empty. ...

Equivalence Relations Induce Partitions

Proof (continued).

For 2) we show $\bigcup_{X \in P} X \subseteq S$ and $\bigcup_{X \in P} X \supseteq S$ separately.

$\subseteq$: Consider an arbitrary $x \in \bigcup_{X \in P} X$. Since x is contained in the union, it must be an element of some $X \in P$. Consider such an X . By the definition of P , there is a $y \in S$ such that $X = [y]_R$.

Since $x \in [y]_R$, it holds that yRx .

As R is a relation over S , this implies that $x \in S$.

$\supseteq$: Consider an arbitrary $x \in S$. Since $x \in [x]_R$ (cf. 1) and $[x]_R \in P$, it holds that $x \in \bigcup_{X \in P} X$

Equivalence Relations Induce Partitions

Proof (continued).

We show 3) by contrapositive:

For all $X, Y \in P$: if $X \cap Y \neq \emptyset$ then $X = Y$.

Let X, Y be two sets from P with $X \cap Y \neq \emptyset$.

Then there is an e with $e \in X \cap Y$ and there are $x, y \in S$ with $X = [x]_R$ and $Y = [y]_R$. Consider such e, x, y .

As $e \in [x]_R$ and $e \in [y]_R$ it holds that xRe and yRe . Since R is symmetric, we get from yRe that eRy . By transitivity, xRe and eRy imply xRy , which by symmetry also gives yRx .

We show $[x]_R \subseteq [y]_R$: consider an arbitrary $z \in [x]_R$. Then xRz . From yRx and xRz , by transitivity we get yRz . This establishes $z \in [y]_R$. As z was chosen arbitrarily, it holds that $[x]_R \subseteq [y]_R$.

Analogously, we can show that $[x]_R \supseteq [y]_R$, so overall $X = Y$. $\square$

Summary

- ▶ We typically encounter equivalence relations when we consider objects as equivalent wrt. some attribute/property.
- ▶ A relation is an **equivalence relation** if it is **reflexive, symmetric and transitive**.
- ▶ A **partition** of a set groups the elements into non-empty subsets.
- ▶ The concepts are closely connected: in principle just different perspectives on the same "situation".