

# Discrete Structures

## A5. Proof Techniques II

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## A5.1 Mathematical Induction

## A5.2 Structural Induction

## A5.3 Excursus: Computer-assisted Theorem Proving

# A5.1 Mathematical Induction

# Proof Techniques

most common proof techniques:

- ▶ direct proof
- ▶ indirect proof (proof by contradiction)
- ▶ contrapositive
- ▶ **mathematical induction**
- ▶ structural induction

# Mathematical Induction

Concrete Mathematics [Graham, Knuth and Patashnik, p. 3]

Mathematical induction proves that

we can climb as high as we like on a ladder,

by proving that we can climb onto the bottom rung (the basis)

and that

from each rung we can climb up to the next one (the step).



# Propositions

Consider a statement on all natural numbers  $n$  with  $n \geq m$ .

- ▶ E.g. “Every natural number  $n \geq 2$  can be written as a product of prime numbers.”
  - ▶  $P(2)$ : “2 can be written as a product of prime numbers.”
  - ▶  $P(3)$ : “3 can be written as a product of prime numbers.”
  - ▶  $P(4)$ : “4 can be written as a product of prime numbers.”
  - ▶ ...
  - ▶  $P(n)$ : “ $n$  can be written as a product of prime numbers.”
  - ▶ For every natural number  $n \geq 2$  proposition  $P(n)$  is true.

**Proposition**  $P(n)$  is a mathematical statement that is defined in terms of natural number  $n$ .

# Mathematical Induction

## Mathematical Induction

Proof (of the truth) of proposition  $P(n)$   
for all natural numbers  $n$  with  $n \geq m$ :

- ▶ **basis**: proof of  $P(m)$
- ▶ **induction hypothesis** (IH):  
suppose that  $P(k)$  is true for all  $k$  with  $m \leq k \leq n$
- ▶ **inductive step**: proof of  $P(n+1)$   
using the induction hypothesis

German: Vollständige Induktion, Induktionsanfang,  
Induktionsannahme oder Induktionsvoraussetzung,  
Induktionsschritt

# Mathematical Induction: Example I

## Theorem

*Every natural number  $n \geq 2$  can be written as a product of prime numbers, i. e.  $n = p_1 \cdot p_2 \cdot \dots \cdot p_m$  with prime numbers  $p_1, \dots, p_m$ .*

## Proof.

Mathematical Induction over  $n$ :

**basis  $n = 2$ :** trivially satisfied, since 2 is prime

**IH:** Every natural number  $k$  with  $2 \leq k \leq n$   
can be written as a product of prime numbers. ...

# Mathematical Induction: Example I

## Theorem

*Every natural number  $n \geq 2$  can be written as a product of prime numbers, i. e.  $n = p_1 \cdot p_2 \cdot \dots \cdot p_m$  with prime numbers  $p_1, \dots, p_m$ .*

## Proof (continued).

inductive step  $n \rightarrow n + 1$ :

- ▶ **Case 1:**  $n + 1$  is a prime number  $\rightsquigarrow$  trivial
- ▶ **Case 2:**  $n + 1$  is not a prime number.

There are natural numbers  $2 \leq q, r \leq n$  with  $n + 1 = q \cdot r$ .

Using the IH shows that there are prime numbers

$q_1, \dots, q_s$  with  $q = q_1 \cdot \dots \cdot q_s$  and

$r_1, \dots, r_t$  with  $r = r_1 \cdot \dots \cdot r_t$ .

Together this means  $n + 1 = q_1 \cdot \dots \cdot q_s \cdot r_1 \cdot \dots \cdot r_t$ .



## Mathematical Induction: Example II

A set is **finite** if it has a finite number of elements.

The **cardinality**  $|S|$  of a finite set  $S$  is the **number of elements** it contains.

### Theorem

*Let  $S$  be a finite set. Then  $|\mathcal{P}(S)| = 2^{|S|}$ .*

What proposition can we use to prove this with mathematical induction?

# Proof by Induction

## Proof.

By induction over  $|S|$ .

**Basis ( $|S| = 0$ ):** Then  $S = \emptyset$  and  $|\mathcal{P}(S)| = |\{\emptyset\}| = 1 = 2^0$ .

**IH:** For all sets  $S$  with  $|S| \leq n$ , it holds that  $|\mathcal{P}(S)| = 2^{|S|}$ .

## Inductive Step ( $n \rightarrow n + 1$ ):

Let  $S'$  be an arbitrary set with  $|S'| = n + 1$  and let  $e$  be an arbitrary element of  $S'$ .

Let further  $S = S' \setminus \{e\}$  and  $X = \{S'' \cup \{e\} \mid S'' \in \mathcal{P}(S)\}$ .

Then  $\mathcal{P}(S') = \mathcal{P}(S) \cup X$ . As  $\mathcal{P}(S)$  and  $X$  are disjoint and  $|X| = |\mathcal{P}(S)|$ , it holds that  $|\mathcal{P}(S')| = 2|\mathcal{P}(S)|$ .

Since  $|S| = n$ , we can use the IH and get

$$|\mathcal{P}(S')| = 2 \cdot 2^{|S|} = 2 \cdot 2^n = 2^{n+1} = 2^{|S'|}.$$



# Weak vs. Strong Induction

- ▶ **Weak induction:** Induction hypothesis only supposes that  $P(k)$  is true for  $k = n$
- ▶ **Strong induction:** Induction hypothesis supposes that  $P(k)$  is true for all  $k \in \mathbb{N}_0$  with  $m \leq k \leq n$ 
  - ▶ also: **complete induction**

Our previous definition corresponds to **strong induction**.

Which of the examples had also worked with weak induction?

# Is Strong Induction More Powerful than Weak Induction?

Are there statements that we can prove with strong induction but not with weak induction?

We can always use a stronger proposition:

- ▶ “Every  $n \in \mathbb{N}_0$  with  $n \geq 2$  can be written as a product of prime numbers.”
- ▶  $P(n)$ : “ $n$  can be written as a product of prime numbers.”
- ▶  $P'(n)$ : “all  $k \in \mathbb{N}_0$  with  $2 \leq k \leq n$  can be written as a product of prime numbers.”

## A5.2 Structural Induction

# Inductively Defined Sets: Examples

## Example (Natural Numbers)

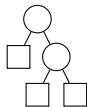
The set  $\mathbb{N}_0$  of natural numbers is inductively defined as follows:

- ▶ 0 is a natural number.
- ▶ If  $n$  is a natural number, then  $n + 1$  is a natural number.

## Example (Binary Tree)

The set  $\mathcal{B}$  of binary trees is inductively defined as follows:

- ▶  $\square$  is a binary tree (a **leaf**)
- ▶ If  $L$  and  $R$  are binary trees, then  $\langle L, \bigcirc, R \rangle$  is a binary tree (with **inner node**  $\bigcirc$ ).



**Implicit statement:** all elements of the set can be constructed by finite application of these rules

German: Binärbaum, Blatt, innerer Knoten

# Inductive Definition of a Set

## Inductive Definition

A set  $M$  can be defined **inductively** by specifying

- ▶ **basic elements** that are contained in  $M$
- ▶ **construction rules** of the form  
“Given some elements of  $M$ , another element of  $M$  can be constructed like this.”

German: Induktive Definition, Basiselemente, Konstruktionsregeln

# Structural Induction

## Structural Induction

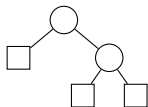
Proof of statement for all elements of an inductively defined set

- ▶ **basis**: proof of the statement for the basic elements
- ▶ **induction hypothesis (IH)**:  
suppose that the statement is true for some elements  $M$
- ▶ **inductive step**: proof of the statement for elements constructed by applying a construction rule to  $M$   
(one inductive step for each construction rule)

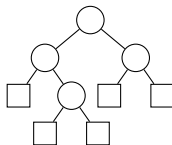
German: Strukturelle Induktion

# Inner Nodes, Leaves and Height of a Binary Tree

by Example



3 leaves  
2 inner nodes  
height 2



5 leaves  
4 inner nodes  
height 3



1 leaf  
0 inner nodes  
height 0

# Inner Nodes and Leaves of a Binary Tree

## Definition (Leaves of a Binary Tree)

The number of **leaves** of a binary tree  $B$ , written  $leaves(B)$ , is defined as follows:

$$leaves(\square) = 1$$

$$leaves(\langle L, \bigcirc, R \rangle) = leaves(L) + leaves(R)$$

## Definition (Inner Nodes of a Binary Tree)

The number of **inner nodes** of a binary tree  $B$ , written  $inner(B)$ , is defined as follows:

$$inner(\square) = 0$$

$$inner(\langle L, \bigcirc, R \rangle) = inner(L) + inner(R) + 1$$

# Structural Induction: Example (1)

## Theorem

*For all binary trees  $B$ :  $inner(B) = leaves(B) - 1$ .*

## Proof.

induction basis:

$$inner(\square) = 0 = 1 - 1 = leaves(\square) - 1$$

$\rightsquigarrow$  statement is true for base case

...

Reminder:  $inner(\square) = 0$  and  $leaves(\square) = 1$

## Structural Induction: Example (2)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree  $\langle L, \bigcirc, R \rangle$ , we may use that it is true for the subtrees  $L$  and  $R$ .

inductive step for  $B = \langle L, \bigcirc, R \rangle$ :

$$\begin{aligned} inner(B) &= inner(L) + inner(R) + 1 \\ &\stackrel{\text{IH}}{=} (leaves(L) - 1) + (leaves(R) - 1) + 1 \\ &= leaves(L) + leaves(R) - 1 = leaves(B) - 1 \end{aligned}$$



Reminder:  $inner(\langle L, \bigcirc, R \rangle) = inner(L) + inner(R) + 1$  and  
 $leaves(\langle L, \bigcirc, R \rangle) = leaves(L) + leaves(R)$

# Structural Induction: Exercise

## Definition (Height of a Binary Tree)

The **height** of a binary tree  $B$ , written  $\text{height}(B)$ , is defined as follows:

$$\text{height}(\square) = 0$$

$$\text{height}(\langle L, \bigcirc, R \rangle) = \max\{\text{height}(L), \text{height}(R)\} + 1$$

**Prove by structural induction:**

## Theorem

*For all binary trees  $B$ :  $\text{leaves}(B) \leq 2^{\text{height}(B)}$ .*

## Example: Tarradiddles

### Example (Tarradiddles)

The set of tarradiddles is inductively defined as follows:

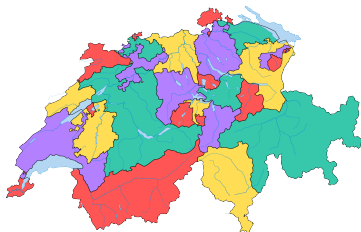
- ▶  $\rightarrow$  is a tarradiddle.
- ▶  $\heartsuit$  is a tarradiddle.
- ▶ If  $x$  and  $y$  are tarradiddles, then  $x\heartsuit\heartsuit y$  is a tarradiddle.
- ▶ If  $x$  and  $y$  are tarradiddles, then  $\heartsuit x\rightarrow y\heartsuit$  is a tarradiddle.

How do you prove with structural induction that every tarradiddle contains an even number of flowers?

## A5.3 Excursus: Computer-assisted Theorem Proving

# Computer-assisted Proofs

- ▶ Computers can help proving theorems.
- ▶ **Computer-aided proofs** have for example been used for proving theorems by exhaustion.
- ▶ Example: **Four color theorem**



# Interactive Theorem Proving

- ▶ On the lowest abstraction level, rigorous mathematical proofs rely on formal logic.
- ▶ On this level, proofs can be automatically verified by computers.
- ▶ Nobody wants to write or read proofs on this level of detail.
- ▶ In Interactive Theorem Proving a human guides the proof and the computer tries to fill in the details.
- ▶ If it succeeds, we can be very confident that the proof is valid.
- ▶ Example theorem provers: Isabelle/HOL, Lean

# Example

LEAN

```
1 import Mathlib.Data.Set.Basic
2
3 theorem diff_empty_subset {α : Type} (A B : Set α) :
4   A \ B = ∅ → A ⊆ B := by
5   intro h
6   by_contra hAB
7   have h_exists : ∃ x, x ∈ A ∧ x ∉ B := Set.not_subset.mp hAB
8   obtain ⟨x, hxA, hxB⟩ := h_exists
9
10  have hxDiff : x ∈ A \ B := ⟨hxA, hxB⟩
11
12  rw [h] at hxDiff
13  simp at hxDiff
```

↪ Demo

<https://live.lean-lang.org/>

# Try It Yourself!

## Natural Number Game

Welcome to the Natural Number Game

*An introduction to mathematical proof.*

In this game, we will build the basic theory of the natural numbers  $\{0, 1, 2, 3, 4, \dots\}$  from scratch. Our first goal is to prove that  $2 + 2 = 4$ . Next we'll prove that  $x + y = y + x$ . And at the end we'll see if we can prove Fermat's Last Theorem. We'll do this by solving levels of a computer puzzle game called Lean.

Read this.

Learning how to use an interactive theorem prover takes time. Tests show that the people who get the most out of this game are those who read the help texts like this one.

To start, click on "Tutorial World".

Note: this is a new Lean 4 version of the game containing several worlds which were not present in the old Lean 3 version. More new worlds such as Strong Induction World, Even/Odd World and Prime Number World are in development; if you want to see their state or even help out, checkout out the [issues in the github repo](#).

More

Click on the three lines in the top right and select "Game Info" for resources, links, and ways to interact with the Lean community.

Natural Number Game

Rules: regular, relaxed, none

| Theorems | Tactics | Definitions |
|----------|---------|-------------|
| apply    | cases   | contrapose  |
| exact    | have    | decide      |
| left     | right   | induction   |
| simp     | rw      | intro       |
| simp_add | system  | simp        |
| use      | tauto   | trivial     |

<https://adam.math.hhu.de/#/g/leanprover-community/nng4>

# Summary

- ▶ **Mathematical induction** is used to prove a proposition  $P$  for all natural numbers  $\geq m$ .
  - ▶ Prove  $P(m)$ .
  - ▶ Make hypothesis that  $P(k)$  is true for  $m \leq k \leq n$ .
  - ▶ Establish  $P(n+1)$  using the hypothesis.
- ▶ **Structural induction** applies the same general concept to prove a proposition  $P$  for all elements of an inductively defined set.