

Discrete Structures

A3. Proofs: Introduction

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A3.1 What is a Proof?

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A **mathematical proof** is

- ▶ a sequence of logical steps
- ▶ starting with one set of statements
- ▶ that comes to the conclusion
that some statement must be true.

What is a **statement**?

Mathematical Statements

Mathematical Statement

A **mathematical statement** is a declarative sentence that is either true or false (but not both).

Examples (some true, some false):

- ▶ Let $p \in \mathbb{N}_0$ be a prime number. Then p is odd.
- ▶ There exists an even prime number.
- ▶ The equation $a^k + b^k = c^k$ has infinitely many solutions with $a, b, c, k \in \mathbb{N}_1$ and $k \geq 2$.

German: Mathematische Aussage

Mathematical Statements: Quantification

Statements often use **quantification**.

- ▶ Universal quantification:
"For all x in set S it holds that $\langle \text{sub-statement on } x \rangle$."
This is **true** if the sub-statement is true for every x in S .
- ▶ Existential quantification:
"There is an x in set S such that $\langle \text{sub-statement on } x \rangle$."
This is **true** if there exists at least one x in S for which the sub-statement is true.

Examples (some true, some false):

- ▶ For all $x \in \mathbb{N}_1$ it holds that $x + 1$ is in $\mathbb{N}_1$.
- ▶ For all $x \in \mathbb{N}_1$ it holds that $x - 1$ is in $\mathbb{N}_1$.
- ▶ There is an $x \in \mathbb{N}_1$ such that $x = \sqrt{x}$.

Mathematical Statements: Preconditions and Conclusions

We can identify **preconditions** and **conclusions**.

"If $\langle \text{preconditions} \rangle$ then $\langle \text{conclusions} \rangle$."

The statement is **true** if the conclusions are true whenever the preconditions are true.

Not every statement has preconditions. Preconditions are often used in universally quantified sub-statements.

Examples (some true, some false):

- ▶ If 4 is a prime number then $2 \cdot 3 = 4$.
- ▶ If n is a prime number with $n > 2$ then n is odd.
- ▶ For all $p \in \mathbb{N}_1$ it holds that if p is a prime number then p is odd.

Different Statements with the same Meaning

The following statements have the same meaning, we just move preconditions into the quantification, make some aspects implicit, and change the structure.

- ▶ For all $p \in \mathbb{N}_1$ it holds that if p is a prime number with $p > 2$ then p is odd.
- ▶ For all prime numbers p it holds that if $p > 2$ then p is odd.
- ▶ Let p be a natural number with $p > 2$. Then p is odd if p is prime.
- ▶ If p is a prime number with $p > 2$ then p is odd.
- ▶ All prime numbers $p > 2$ are odd.

A single mathematical statement can be expressed in different ways, as long as the meaning stays the same.

Like paraphrasing a sentence in everyday language.

On what Statements can we Build the Proof?

A mathematical proof is

- ▶ a sequence of logical steps
- ▶ **starting with one set of statements**
- ▶ that comes to the conclusion that some statement must be true.

We can use:

- ▶ **axioms**: statements that are assumed to always be true in the current context
- ▶ **theorems** and **lemmas**: statements that were already proven
 - ▶ lemma: an intermediate tool
 - ▶ theorem: itself a relevant result
- ▶ **premises**: assumptions we make to see what consequences they have

German: Axiom, Theorem/Satz, Lemma, Prämisse/Annahme

What is a Logical Step?

A mathematical proof is

- ▶ **a sequence of logical steps**
- ▶ starting with one set of statements
- ▶ that comes to the conclusion that some statement must be true.

Each step **directly follows**

- ▶ from the axioms,
- ▶ premises,
- ▶ previously proven statements and
- ▶ the preconditions of the statement we want to prove.

For a formal definition, we would need formal logics.

The Role of Definitions

Definition

A **set** is an unordered collection of distinct objects.

The objects in a set are called the **elements** of the set. A set is said to **contain** its elements.

We write $x \in S$ to indicate that x is an element of set S , and $x \notin S$ to indicate that S does not contain x .

The set that does not contain any objects is the **empty set** $\emptyset$.

- ▶ A definition introduces an abbreviation.
- ▶ Whenever we say “set”, we could instead say “an unordered collection of distinct objects” and vice versa.
- ▶ Definitions can also introduce notation.

German: Definition

Disproofs

- ▶ A **disproof** (**refutation**) shows that a given mathematical statement is **false** by giving an example where the preconditions are true, but the conclusion is false.
- ▶ This requires deriving, in a sequence of proof steps, the opposite (negation) of the conclusion.

Example (False statement)

“If $p \in \mathbb{N}_0$ is a prime number then p is odd.”

Refutation.

Consider natural number 2 as a counter example. It is prime because it has exactly 2 divisors, 1 and itself. It is not odd, because it is divisible by 2. □

German: Widerlegung

A Word on Style

A proof should help the reader to see why the result must be true.

- ▶ A proof should be easy to follow.
- ▶ Omit unnecessary information.
- ▶ Move self-contained parts into separate lemmas.
- ▶ In complicated proofs, reveal the overall structure in advance.
- ▶ Have a clear line of argument.

→ Writing a proof is like writing an essay.

Recommended reading (ADAM additional resources):

- ▶ “Some Remarks on Writing Mathematical Proofs” (John M. Lee)
- ▶ “§1. Minicourse on technical writing” of “Mathematical Writing” (Donald E. Knuth, Tracy Larrabee, and Paul M. Roberts)

Summary

A proof should convince the reader by **logical steps** of the truth of some mathematical statement.