

Discrete Structures

A2. Sets: Foundations

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Sets

Important Building Blocks of Discrete Mathematics

- sets
- relations
- functions

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We cover some foundations on sets already now because we will use them for illustrating proof techniques.

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German: Menge

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- **unordered**: no notion of a “first” or “second” object,
e. g. $\{Alice, Bob, Charlie\} = \{Charlie, Bob, Alice\}$
- **distinct**: each object contained **at most once**,
e. g. $\{Alice, Bob, Charlie\} = \{Alice, Charlie, Bob, Alice\}$

German: Menge

Notation

■ Specification of sets

- **explicit**, listing all elements, e. g. $A = \{1, 2, 3\}$
- **implicit** with **set-builder notation**,
specifying a **property** characterizing all elements,
e. g. $A = \{x \mid x \in \mathbb{N}_0 \text{ and } 1 \leq x \leq 3\}$,
 $B = \{n^2 \mid n \in \mathbb{N}_0\}$
- **implicit**, as a **sequence with dots**,
e. g. $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
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Question: Is it true that $1 \in \{\{1, 2\}, 3\}$?

German: Element, leere Menge

Special Sets

- Natural numbers

$$\mathbb{N}_0 = \{0, 1, 2, \dots\}$$

$$\mathbb{N}_1 = \{1, 2, \dots\}$$

German: Natürliche ($\mathbb{N}_0$), ganze ($\mathbb{Z}$), rationale ($\mathbb{Q}$), reelle ($\mathbb{R}$) Zahlen

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- **Real numbers** $\mathbb{R} = (-\infty, \infty)$

Why do we use interval notation?

Why didn't we introduce it before?

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Questions



Questions?

Relations on Sets

Equality

Definition (Axiom of Extensionality)

Two sets A and B are **equal** (written $A = B$) if every element of A is an element of B and vice versa.

Two sets are equal if they contain the same elements.

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Two sets are equal if they contain the same elements.

We write $A \neq B$ to indicate that A and B are **not** equal.

Subsets and Supersets

- $A \subseteq B$: A is a **subset** of B ,
i. e., every element of A is an element of B
- $A \subset B$: A is a **strict subset** of B ,
i. e., $A \subseteq B$ and $A \neq B$.
- $A \supseteq B$: A is a **superset** of B if $B \subseteq A$.
- $A \supset B$: A is a **strict superset** of B if $B \subset A$.

German: Teilmenge, echte Teilmenge, Obermenge, echte Obermenge

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We write $A \not\subseteq B$ to indicate that A is **not** a subset of B .

Analogously: $\not\subset$, $\not\supseteq$, $\not\supset$

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Power Set

Definition (Power Set)

The **power set** $\mathcal{P}(S)$ of a set S is the set of all subsets of S .
That is,

$$\mathcal{P}(S) = \{M \mid M \subseteq S\}.$$

Example: $\mathcal{P}(\{a, b\}) =$

German: Potenzmenge

Questions



Questions?

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- **intersection** $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



If $A \cap B = \emptyset$ then A and B are **disjoint**.

German: Schnitt, disjunkt

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- **set difference** $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



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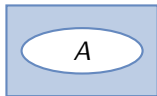
- **union** $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



- **set difference** $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



- **complement** $\bar{A} = B \setminus A$, where $A \subseteq B$ and B is the set of all considered objects (in a given context)



German: Schnitt, disjunkt, Vereinigung,
Differenz, Komplement

Properties of Set Operations: Commutativity

Theorem (Commutativity of $\cup$ and $\cap$)

For all sets A and B it holds that

- $A \cup B = B \cup A$ and
- $A \cap B = B \cap A$.

German: Kommutativität

Properties of Set Operations: Commutativity

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For all sets A and B it holds that

- $A \cup B = B \cup A$ and
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Question: Is the set difference also commutative,
i. e. is $A \setminus B = B \setminus A$ for all sets A and B ?

German: Kommutativität

Properties of Set Operations: Associativity

Theorem (Associativity of $\cup$ and $\cap$)

For all sets A, B and C it holds that

- $(A \cup B) \cup C = A \cup (B \cup C)$ and
- $(A \cap B) \cap C = A \cap (B \cap C)$.

German: Assoziativität

Properties of Set Operations: Distributivity

Theorem (Union distributes over intersection and vice versa)

For all sets A, B and C it holds that

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

German: Distributivität

Properties of Set Operations: De Morgan's Law



Augustus De Morgan

British mathematician (1806-1871)

Theorem (De Morgan's Law)

For all sets A and B it holds that

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and
- $\overline{A \cap B} = \overline{A} \cup \overline{B}.$

Questions



Questions?

Russell's Paradox

Excursus: Barber Paradox

Barber Paradox

In a town there is only one barber, who is male.
The barber shaves all men in the town,
and only those, who do not shave themselves.



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Who shaves the barber?



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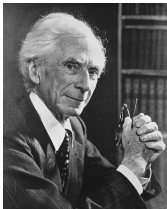
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We can exploit the self-reference to derive a contradiction.

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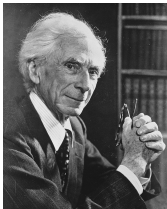


Bertrand Russell

Question

Is the collection of all sets that do not contain themselves as a member a set?

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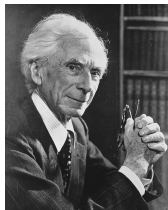
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Is the collection of all sets that do not contain themselves as a member a set?

Is $S = \{M \mid M \text{ is a set and } M \notin M\}$ a set?

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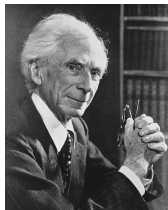
Assume that S is a set.

If $S \notin S$ then $S \in S \rightsquigarrow$ Contradiction

If $S \in S$ then $S \notin S \rightsquigarrow$ Contradiction

Hence, there is no such set S .

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Hence, there is no such set S .

→ Not every property used in set-builder notation defines a set.



<https://existentialcomics.com/comic/98>

Questions



Questions?

Summary

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- Sets are unordered collections of distinct objects.
- Important set relations: equality ($=$), subset ($\subseteq$), superset ($\supseteq$) and strict variants ($\subset$ and $\supset$)
- The power set of a set S is the set of all subsets of S .
- Important set operations are intersection, union, set difference and complement.
 - Union and intersection are commutative and associative.
 - Union distributes over intersection and vice versa.
 - De Morgan's law for complement of union or intersection.