Planning and Optimization
F10. Network Flow Heuristics

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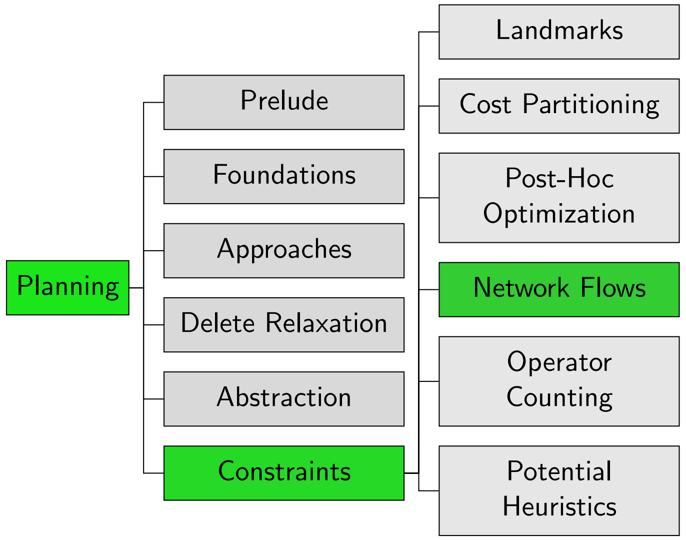


F10.1 Introduction

Reminder: SAS$^+$ Planning Tasks

For a SAS$^+$ planning task $\Pi = \langle V, I, O, \gamma\rangle$:

- V is a set of finite-domain state variables,
- Each atom has the form $v = d$ with $v \in V, d \in \text{dom}(v)$.
- Operator preconditions and the goal formula γ are satisfiable conjunctions of atoms.
- Operator effects are conflict-free conjunctions of atomic effects of the form $v_1 := d_1 \wedge \cdots \wedge v_n := d_n$.

Example Task (1)

- One package, two trucks, two locations
- Variables:
  - *pos-p* with $\text{dom}(\textit{pos-p}) = \{loc_1, loc_2, t_1, t_2\}$
  - *pos-t-i* with $\text{dom}(\textit{pos-t-i}) = \{loc_1, loc_2\}$ for $i \in \{1, 2\}$
- The package is at location 1 and the trucks at location 2,
  - $I = \{\textit{pos-p} \mapsto loc_1, \textit{pos-t-1} \mapsto loc_2, \textit{pos-t-2} \mapsto loc_2)$
- The goal is to have the package at location 2 and truck 1 at location 1.
  - $\gamma = (\textit{pos-p} = loc_2) \wedge (\textit{pos-t-1} = loc_1)$

Example Task (2)

- Operators: for $i, j, k \in \{1, 2\}$:

$$\begin{aligned} load(t_i, loc_j) &= \langle \textit{pos-t-i} = loc_j \wedge \textit{pos-p} = loc_j, \\ &\quad \textit{pos-p} := t_i, 1\rangle \\ unload(t_i, loc_j) &= \langle \textit{pos-t-i} = loc_j \wedge \textit{pos-p} = t_i, \\ &\quad \textit{pos-p} := loc_j, 1\rangle \\ drive(t_i, loc_j, loc_k) &= \langle \textit{pos-t-i} = loc_j, \\ &\quad \textit{pos-t-i} := loc_k, 1\rangle \end{aligned}$$

Example Task: Observations

Consider some atoms of the example task:

- $\textit{pos-p} = loc_1$ initially true and must be false in the goal
  ▷ at location 1 the package must be loaded once more than it is unloaded.
- $\textit{pos-p} = loc_2$ initially false and must be true in the goal
  ▷ at location 2 the package must be unloaded once more than it is loaded.
- $\textit{pos-p} = t_1$ initially false and must be false in the goal
  ▷ same number of load and unload actions for truck 1.

Can we derive a heuristic from this kind of information?

Example: Flow Constraints

Let π be some arbitrary plan for the example task and let $\#o$ denote the number of occurrences of operator o in π.
Then the following holds:

- $\mathit{pos\text{-}p} = \mathit{loc}_1$ initially true and must be false in the goal
  ▷ at location 1 the package must be loaded once more than it is unloaded.
  $\#\mathit{load}(t_1, \mathit{loc}_1) + \#\mathit{load}(t_2, \mathit{loc}_1) = 1 + \#\mathit{unload}(t_1, \mathit{loc}_1) + \#\mathit{unload}(t_2, \mathit{loc}_1)$
- $\mathit{pos\text{-}p} = t_1$ initially false and must be false in the goal
  ▷ same number of load and unload actions for truck 1.
  $\#\mathit{unload}(t_1, \mathit{loc}_1) + \#\mathit{unload}(t_1, \mathit{loc}_2) = \#\mathit{load}(t_1, \mathit{loc}_1) + \#\mathit{load}(t_1, \mathit{loc}_2)$

Network Flow Heuristics: General Idea

- Formulate flow constraints for each atom.
- These are satisfied by every plan of the task.
- The cost of a plan is $\sum_{o \in O} \mathit{cost}(o)\#o$
- The objective value of an integer program that minimizes this cost subject to the flow constraints is a lower bound on the plan cost (i.e., an admissible heuristic estimate).
- As solving the IP is NP-hard, we solve the LP relaxation instead.

How do we get the flow constraints?

How to Derive Flow Constraints?

- The constraints formulate how often an atom can be produced or consumed.
- "Produced" (resp. "consumed") means that the atom is false (resp. true) before an operator application and true (resp. false) in the successor state.
- For general SAS$^+$ operators, this depends on the state where the operator is applied: effect $v := d$ only produces $v = d$ if the operator is applied in a state s with $s(v) \neq d$.
- For general SAS$^+$ tasks, the goal does not have to specify a value for every variable.
- All this makes the definition of flow constraints somewhat involved and in general such constraints are inequalitites.

Good news: easy for tasks in transition normal form

F10.2 Transition Normal Form

Variables Occurring in Conditions and Effects

- Many algorithmic problems for SAS$^+$ planning tasks become simpler when we can make two further restrictions.
- These are related to the variables that occur in conditions and effects of the task.

Definition ($vars(\varphi)$, $vars(e)$)

For a logical formula φ over finite-domain variables V, $vars(\varphi)$ denotes the set of finite-domain variables occurring in φ.

For an effect e over finite-domain variables V, $vars(e)$ denotes the set of finite-domain variables occurring in e.

Transition Normal Form

Definition (Transition Normal Form)

A SAS$^+$ planning task $\Pi = \langle V, I, O, \gamma \rangle$ is in transition normal form (TNF) if

- for all $o \in O$, $vars(pre(o)) = vars(eff(o))$, and
- $vars(\gamma) = V$.

In words, an operator in TNF must mention the same variables in the precondition and effect, and a goal in TNF must mention all variables (= specify exactly one goal state).

Converting Operators to TNF: Violations

There are two ways in which an operator o can violate TNF:

- There exists a variable $v \in vars(pre(o)) \setminus vars(eff(o))$.
- There exists a variable $v \in vars(eff(o)) \setminus vars(pre(o))$.

The first case is easy to address: if $v = d$ is a precondition with no effect on v, just add the effect $v := d$.

The second case is more difficult: if we have the effect $v := d$ but no precondition on v, how can we add a precondition on v without changing the meaning of the operator?

Converting Operators to TNF: Multiplying Out

Solution 1: multiplying out

1. While there exists an operator o and a variable $v \in vars(eff(o))$ with $v \notin vars(pre(o))$:
   - For each $d \in \mathrm{dom}(v)$, add a new operator that is like o but with the additional precondition $v = d$.
   - Remove the original operator.
2. Repeat the previous step until no more such variables exist.

Problem:

- If an operator o has n such variables, each with k values in its domain, this introduces k^n variants of o.
- Hence, this is an exponential transformation.

Converting Operators to TNF: Auxiliary Values

Solution 2: auxiliary values

1. For every variable v, add a new auxiliary value u to its domain.
2. For every variable v and value $d \in \mathrm{dom}(v) \setminus \{\mathrm{u}\}$, add a new operator to change the value of v from d to u at no cost: $\langle v = d, v := \mathrm{u}, 0\rangle$.
3. For all operators o and all variables $v \in \mathit{vars}(\mathit{eff}(o)) \setminus \mathit{vars}(\mathit{pre}(o))$, add the precondition $v = \mathrm{u}$ to $\mathit{pre}(o)$.

Properties:

- Transformation can be computed in linear time.
- Due to the auxiliary values, there are new states and transitions in the induced transition system, but all path costs between original states remain the same.

Converting Goals to TNF

- The auxiliary value idea can also be used to convert the goal γ to TNF.
- For every variable $v \notin \mathit{vars}(\gamma)$, add the condition $v = \mathrm{u}$ to γ.

With these ideas, every SAS$^+$ planning task can be converted into transition normal form in linear time.

TNF for Example Task (1)

The example task is not in transition normal form:

- Load and unload operators have preconditions on the position of some truck but no effect on this variable.
- The goal does not specify a value for variable *pos-t-2*.

TNF for Example Task (2)

Operators in transition normal form: for $i, j, k \in \{1, 2\}$:

$$\begin{aligned}
\mathit{load}(t_i, \mathit{loc}_j) &= \langle \mathit{pos\text{-}t\text{-}i} = \mathit{loc}_j \land \mathit{pos\text{-}p} = \mathit{loc}_j, \\
&\qquad \mathit{pos\text{-}p} := t_i \land \mathit{pos\text{-}t\text{-}i} := \mathit{loc}_j, 1\rangle \\
\mathit{unload}(t_i, \mathit{loc}_j) &= \langle \mathit{pos\text{-}t\text{-}i} = \mathit{loc}_j \land \mathit{pos\text{-}p} = t_i, \\
&\qquad \mathit{pos\text{-}p} := \mathit{loc}_j \land \mathit{pos\text{-}t\text{-}i} := \mathit{loc}_j, 1\rangle \\
\mathit{drive}(t_i, \mathit{loc}_j, \mathit{loc}_k) &= \langle \mathit{pos\text{-}t\text{-}i} = \mathit{loc}_j, \\
&\qquad \mathit{pos\text{-}t\text{-}i} := \mathit{loc}_k, 1\rangle
\end{aligned}$$

TNF for Example Task (3)

To bring the goal in normal form,

- add an additional value **u** to dom(*pos-t-2*)
- add zero-cost operators
  $o_1 = \langle \mathit{pos\text{-}t\text{-}2} = \mathit{loc}_1, \mathit{pos\text{-}t\text{-}2} := \mathbf{u}, 0\rangle$ and
  $o_2 = \langle \mathit{pos\text{-}t\text{-}2} = \mathit{loc}_2, \mathit{pos\text{-}t\text{-}2} := \mathbf{u}, 0\rangle$
- Add $\mathit{pos\text{-}t\text{-}2} = \mathbf{u}$ to the goal:
  $\gamma = (\mathit{pos\text{-}p} = \mathit{loc}_2) \land (\mathit{pos\text{-}t\text{-}1} = \mathit{loc}_1) \land (\mathit{pos\text{-}t\text{-}2} = \mathbf{u})$

F10.3 Flow Heuristic

Notation

- In SAS$^+$ tasks, states are variable assignments, conditions are conjunctions over atoms, and effects are conjunctions of atomic effects.
- In the following, we use a unifying notation to express that an atom is true in a state/entailed by a condition/ made true by an effect.
- For state s, we write $(v = d) \in s$ to express that $s(v) = d$.
- For a conjunction of atoms φ, we write $(v = d) \in \varphi$ to express that φ has a conjunct $v = d$ (or alternatively $\varphi \models v = d$).
- For effect e, we write $(v = d) \in e$ to express that e contains the atomic effect $v := d$.

Flow Constraints (1)

A flow constraint for an atom relates how often it can be produced to how often it can be consumed.

Let o be an operator in transition normal form. Then:

- o produces atom a iff $a \in \mathit{eff}(o)$ and $a \notin \mathit{pre}(o)$.
- o consumes atom a iff $a \in \mathit{pre}(o)$ and $a \notin \mathit{eff}(o)$.
- Otherwise o is neutral wrt. atom a.

$\rightsquigarrow$ State-independent

Flow Constraints (2)

A flow constraint for an atom relates how often it can be produced to how often it can be consumed.

The constraint depends on the current state s and the goal γ.
If γ mentions all variables (as in TNF), the following holds:

- If $a \in s$ and $a \in \gamma$ then atom a must be equally often produced and consumed.
- Analogously for $a \notin s$ and $a \notin \gamma$.
- If $a \in s$ and $a \notin \gamma$ then a must be consumed once more than it is produced.
- If $a \notin s$ and $a \in \gamma$ then a must be produced once more than it is consumed.

Iverson Bracket

The dependency on the current state and the goal can concisely be expressed with Iverson brackets:

Definition (Iverson Bracket)

Let P be a logical proposition (= some statement that can be evaluated to true or false). Then

$$[P] = \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false.} \end{cases}$$

Example: $[2 \neq 3] = 1$

Flow Constraints (3)

Definition (Flow Constraint)

Let $\Pi = \langle V, I, O, \gamma \rangle$ be a task in transition normal form.
The flow constraint for atom a in state s is

$$[a \in s] + \sum_{o \in O : a \in \mathit{eff}(o)} \mathsf{Count}_o = [a \in \gamma] + \sum_{o \in O : a \in \mathit{pre}(o)} \mathsf{Count}_o$$

- Count_o is an LP variable for the number of occurrences of operator o.
- Neutral operators either appear on both sides or on none.

Flow Heuristic

Definition (Flow Heuristic)

Let $\Pi = \langle V, I, O, \gamma \rangle$ be a SAS$^+$ task in transition normal form and let $A = \{(v = d) \mid v \in V, d \in \mathit{dom}(v)\}$ be the set of atoms of Π.

The flow heuristic $h^{\text{flow}}(s)$ is the objective value of the following LP or ∞ if the LP is infeasible:

$$\text{minimize} \quad \sum_{o \in O} \mathit{cost}(o) \cdot \mathsf{Count}_o \quad \text{subject to}$$

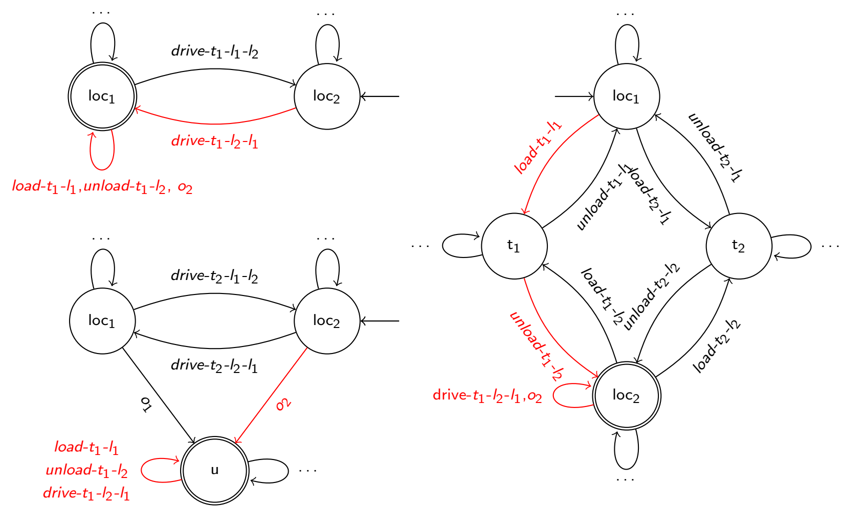
$$[a \in s] + \sum_{o \in O : a \in \mathit{eff}(o)} \mathsf{Count}_o = [a \in \gamma] + \sum_{o \in O : a \in \mathit{pre}(o)} \mathsf{Count}_o \quad \text{for all } a \in A$$

$$\mathsf{Count}_o \geq 0 \quad \text{for all } o \in O$$

Flow Heuristic on Example Task

$\rightsquigarrow$ Demo

Visualization of Flow in Example Task



Flow Heuristic: Properties (1)

Theorem

The flow heuristic h^{flow} is goal-aware, safe, consistent and admissible.

Proof Sketch.

It suffices to prove goal-awareness and consistency.

Goal-awareness: If $s \models \gamma$ then $\mathrm{Count}_o = 0$ for all $o \in O$ is feasible and the objective function has value 0. As $\mathrm{Count}_o \geq 0$ for all variables and operator costs are nonnegative, the objective value cannot be smaller. . . .

Flow Heuristic: Properties (2)

Proof Sketch (continued).

Consistency: Let o be an operator that is applicable in state s and let $s' = s[\![o]\!]$.

Increasing Count_o by one in an optimal feasible assignment for the LP for state s' yields a feasible assignment for the LP for state s, where the objective function is $h^{\mathrm{flow}}(s') + cost(o)$.

This is an upper bound on $h^{\mathrm{flow}}(s)$, so in total $h^{\mathrm{flow}}(s) \leq h^{\mathrm{flow}}(s') + cost(o)$. $\square$

F10.4 Summary

Summary

- A flow constraint for an atom describes how the number of producing operator applications is linked to the number of consuming operator applications.
- The flow heuristic computes a lower bound on the cost of each operator sequence that satisfies these constraints for all atoms.
- The flow heuristic only considers the number of occurrences of each operator, but ignores their order.