Planning and Optimization

F9. Post-hoc Optimization

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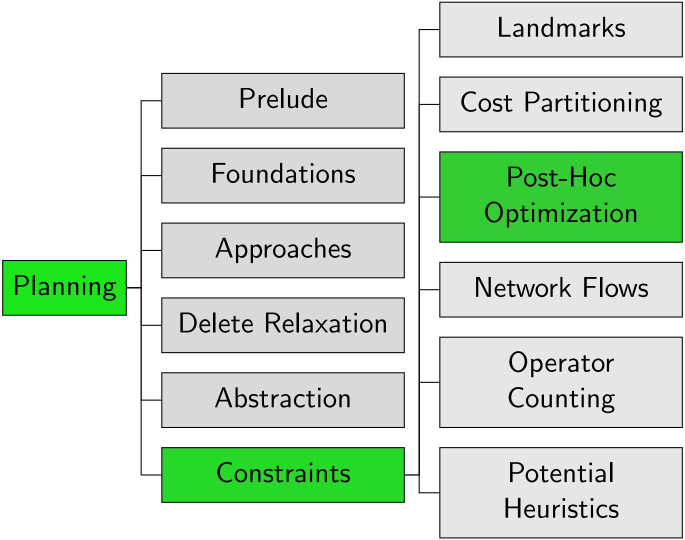
Content of the Course



F9.1 Introduction

Example Task (1)

Example (Example Task)

SAS$^+$ task $\Pi = \langle V, I, O, \gamma \rangle$ with

- $V = \{A, B, C\}$ with $\mathrm{dom}(v) = \{0, 1, 2, 3, 4\}$ for all $v \in V$
- $I = \{A \mapsto 0, B \mapsto 0, C \mapsto 0\}$
- $O = \{inc_x^v \mid v \in V, x \in \{0, 1, 2\}\} \cup \{jump^v \mid v \in V\}$
  - $inc_x^v = \langle v = x, v := x + 1, 1 \rangle$
  - $jump^v = \langle \bigwedge_{v' \in V : v' \neq v} v' = 4, v := 3, 1 \rangle$
- $\gamma = A = 3 \land B = 3 \land C = 3$

- Each optimal plan consists of three increment operators for each variable $\rightsquigarrow h^*(I) = 9$
- Each operator affects only one variable.

Example Task (2)

- In projections on single variables we can reach the goal with a *jump* operator: $h^{\{A\}}(I) = h^{\{B\}}(I) = h^{\{C\}}(I) = 1$.
- In projections on more variables, we need for each variable three applications of increment operators to reach the abstract goal from the abstract initial state: $h^{\{A,B\}}(I) = h^{\{A,C\}}(I) = h^{\{B,C\}}(I) = 6$

Example (Canonical Heuristic)

$\mathcal{C} = \{\{A\}, \{B\}, \{C\}, \{A, B\}, \{A, C\}, \{B, C\}\}$

$$h^{\mathcal{C}}(s) = \max\{h^{\{A\}}(s) + h^{\{B\}}(s) + h^{\{C\}}(s), h^{\{A\}}(s) + h^{\{B,C\}}(s), \\ h^{\{B\}}(s) + h^{\{A,C\}}(s), h^{\{C\}}(s) + h^{\{A,B\}}(s)\}$$

$h^{\mathcal{C}}(I) = 7$

Post-hoc Optimization Heuristic: Idea

Consider the example task:

- *type-v* operator: operator modifying variable v
- $h^{\{A,B\}} = 6$
  $\Rightarrow$ in any plan operators of type A or B incur at least cost 6.
- $h^{\{A,C\}} = 6$
  $\Rightarrow$ in any plan operators of type A or C incur at least cost 6.
- $h^{\{B,C\}} = 6$
  $\Rightarrow$ in any plan operators of type B or C incur at least cost 6.
- $\Rightarrow$ any plan has at least cost ???.
- (let's use linear programming. . .)
- $\Rightarrow$ any plan has at least cost 9.

Can we generalize this kind of reasoning?

F9.2 Post-hoc Optimization

Post-hoc Optimization

The heuristic that generalizes this kind of reasoning is the Post-hoc Optimization Heuristic (PhO)

- can be computed for any kind of heuristic ...
- ... as long as we are able to determine relevance of operators
- if in doubt, it's always safe to assume an operator is relevant for a heuristic
- but for PhO to work well, it's important that the set of relevant operators is as small as possible

Operator Relevance in Abstractions

Definition (Reminder: Affecting Transition Labels)

Let $\mathcal{T}$ be a transition system, and let ℓ be one of its labels.

We say that ℓ affects $\mathcal{T}$ if $\mathcal{T}$ has a transition $s \xrightarrow{\ell} t$ with $s \neq t$.

Definition (Operator Relevance in Abstractions)

An operator o is relevant for an abstraction α if o affects $\mathcal{T}^\alpha$.

We can efficiently determine operator relevance for abstractions.

Linear Program (1)

For a given set of abstractions $\{\alpha_1, \dots, \alpha_n\}$, we construct a linear program:

- variable X_o for each operator $o \in O$
- intuitively, X_o is cost incurred by operator o
- abstraction heuristics are admissible

$$\sum_{o \in O} X_o \geq h^{\alpha}(s) \quad \text{for } \alpha \in \{\alpha_1, \dots, \alpha_n\}$$

- can tighten these constraints to

$$\sum_{o \in O : o \text{ relevant for } \alpha} X_o \geq h^{\alpha}(s) \quad \text{for } \alpha \in \{\alpha_1, \dots, \alpha_n\}$$

Linear Program (2)

For set of abstractions $\{\alpha_1, \dots, \alpha_n\}$:

Variables

Non-negative variables X_o for all operators $o \in O$

Objective

Minimize $\sum_{o \in O} X_o$

Subject to

$$
\begin{aligned}
\sum_{o \in O : o \text{ relevant for } \alpha} X_o &\geq h^{\alpha}(s) && \text{for } \alpha \in \{\alpha_1, \dots, \alpha_n\} \\
X_o &\geq 0 && \text{for all } o \in O
\end{aligned}
$$

Simplifying the LP

- Reduce the size of the LP by aggregating variables which always occur together in constraints.
- Happens if several operators are relevant for exactly the same heuristics.
- Partitioning $O/\!\sim$ induced by this equivalence relation
- One variable $X_{[o]}$ for each $[o] \in O/\!\sim$

Example

Example

- only operators o_1, o_2, o_3 and o_4 are relevant for h_1 and $h_1(s_0) = 11$
- only operators o_3, o_4, o_5 and o_6 are relevant for h_2 and $h_2(s_0) = 11$
- only operators o_1, o_2 and o_6 are relevant for h_3 and $h_3(s_0) = 8$

Which operators are relevant for exactly the same heuristics?
What is the resulting partitioning?

Answer: $o_1 \sim o_2$ and $o_3 \sim o_4$
$\Rightarrow O/\!\sim\; = \{[o_1], [o_3], [o_5], [o_6]\}$

Simplifying the LP: Example

LP before aggregation

Variables

Non-negative variable $X_1, \dots, X_6$
for operators $o_1, \dots, o_6$

$$
\text{Minimize} \quad X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \quad \text{subject to}
$$

$$
\begin{aligned}
X_1 + X_2 + X_3 + X_4 \phantom{{}+ X_5 + X_6} &\geq 11 \\
X_3 + X_4 + X_5 + X_6 &\geq 11 \\
X_1 + X_2 \phantom{{}+ X_3 + X_4 + X_5} + X_6 &\geq 8 \\
X_i &\geq 0 \quad \text{for } i \in \{1, \dots, 6\}
\end{aligned}
$$

Simplifying the LP: Example

LP after aggregation

Variables

Non-negative variable $X_{[1]}, X_{[3]}, X_{[5]}, X_{[6]}$
for equivalence classes $[o_1], [o_3], [o_5], [o_6]$

$$\text{Minimize} \quad X_{[1]} + X_{[3]} + X_{[5]} + X_{[6]} \quad \text{subject to}$$

$$\begin{aligned}
X_{[1]} + X_{[3]} \qquad\qquad\qquad\qquad &\geq 11 \\
X_{[3]} + X_{[5]} + X_{[6]} &\geq 11 \\
X_{[1]} + \qquad\qquad\qquad + X_{[6]} &\geq 8 \\
X_i &\geq 0 \quad \text{for } i \in \{[1],[3],[5],[6]\}
\end{aligned}$$

PhO Heuristic

Definition (Post-hoc Optimization Heuristic)

The post-hoc optimization heuristic $h^{\mathrm{PhO}}_{\{\alpha_1,\dots,\alpha_n\}}$ for abstractions $\alpha_1, \dots, \alpha_n$ is the objective value of the following linear program:

$$\text{Minimize} \sum_{[o] \in O/\sim} X_{[o]} \text{ subject to}$$

$$\begin{aligned} \sum_{[o] \in O/\sim : o \text{ relevant for } \alpha} X_{[o]} &\geq h^{\alpha}(s) && \text{for all } \alpha \in \{\alpha_1, \dots, \alpha_n\} \\ X_{[o]} &\geq 0 && \text{for all } [o] \in O/\sim, \end{aligned}$$

where $o \sim o'$ iff o and o' are relevant for exactly the same abstractions in $\alpha_1, \dots, \alpha_n$.

PhO Heuristic

h^{PhO}

1. Precompute all abstraction heuristics $h^{\alpha_1}, \dots, h^{\alpha_n}$.
2. Create LP for initial state s_0.
3. For each new state s:
   - Look up $h^{\alpha}(s)$ for all $\alpha \in \{\alpha_1, \dots, \alpha_n\}$.
   - Adjust LP by replacing bounds with the $h^{\alpha}(s)$ values.

Post-hoc Optimization Heuristic: Admissibility

Theorem (Admissibility)

The post-hoc optimization heuristic is admissible.

Proof.

Let Π be a planning task and $\{\alpha_1, \dots, \alpha_n\}$ be a set of abstractions. We show that there is a feasible variable assignment with objective value equal to the cost of an optimal plan.
Let π be an optimal plan for state s and let $cost_\pi(O')$ be the cost incurred by operators from $O' \subseteq O$ in π.

Setting each $X_{[o]}$ to $cost_\pi([o])$ is a feasible variable assignment:
Constraints $X_{[o]} \geq 0$ are satisfied. ...

Post-hoc Optimization Heuristic: Admissibility

Theorem (Admissibility)

The post-hoc optimization heuristic is admissible.

Proof (continued).

For each $\alpha \in \{\alpha_1, \dots, \alpha_n\}$, π is a solution in the abstract transition system and the sum in the corresponding constraint equals the cost of the state-changing abstract state transitions (i.e.. not accounting for self-loops). As $h^\alpha(s)$ corresponds to the cost of an optimal solution in the abstraction, the inequality holds.

For this assignment, the objective function has value $h^*(s)$ (cost of π), so the objective value of the LP is admissible. □

F9.3 Comparison

Combining Estimates from Abstraction Heuristics

- Post-Hoc optimization combines multiple admissible heuristic estimates into one.
- We have already heard of two other such approaches for abstraction heuristics,
  - the canonical heuristic (for PDBs), and
  - optimal cost partitioning (not covered in detail).
- How does PhO compare to these?

What about Optimal Cost Partitioning for Abstractions?

Optimal cost partitioning for abstractions...

- ...uses a state-specific LP to find the best possible cost partitioning, and sums up the heuristic estimates.
- ...dominates the canonical heuristic, i.e. for the same pattern collection, it never gives lower estimates than $h^{\mathcal{C}}$.
- ...is very expensive to compute (recomputing all abstract goal distances in every state).

PhO: Dual Linear Program

For set of abstractions $\{\alpha_1, \dots, \alpha_n\}$:

Variables

Y_α for each abstraction $\alpha \in \{\alpha_1, \dots, \alpha_n\}$

Objective

Maximize $\sum_{\alpha \in \{\alpha_1, \dots, \alpha_n\}} h^\alpha(s) Y_\alpha$

Subject to

$$\sum_{\alpha \in \{\alpha_1, \dots, \alpha_n\} : o \text{ relevant for } \alpha} Y_\alpha \leq 1 \quad \text{for all } [o] \in O/\sim$$

$$Y_\alpha \geq 0 \quad \text{for all } \alpha \in \{\alpha_1, \dots, \alpha_n\}$$

We compute a state-specific cost partitioning that can only scale the operator costs within each heuristic by a factor $0 \leq Y_\alpha \leq 1$.

Relation to Optimal Cost Partitioning

Theorem

Optimal cost partitioning dominates post-hoc optimization.

Proof Sketch.

Consider a feasible assignment $\langle Y_{\alpha_1}, \dots, Y_{\alpha_n} \rangle$ for the variables of the dual LP for PhO.

Its objective value is equivalent to the cost-partitioning heuristic for the same abstractions with cost partitioning $\langle Y_{\alpha_1} \mathit{cost}, \dots, Y_{\alpha_n} \mathit{cost} \rangle$.

Relation to Canonical Heuristic

Theorem

Consider the dual D of the LP solved by the post-hoc optimization heuristic in state s for a given set of abstractions. If we restrict the variables in D to integers, the objective value is the canonical heuristic value $h^{\mathcal{C}}(s)$.

Corollary

The post-hoc optimization heuristic dominates the canonical heuristic for the same set of abstractions.

h^{PhO} vs $h^{\mathcal{C}}$

- For the canonical heuristic, we need to find all maximal cliques, which is an NP-hard problem.
- The post-hoc optimization heuristic dominates the canonical heuristic and can be computed in polynomial time.
- The post-hoc optimization heuristic solves an LP in each state.
- With post-hoc optimization, a large number of small patterns works well.

F9.4 Summary

Summary

- Post-hoc optimization heuristic constraints express admissibility of heuristics
- exploits (ir-)relevance of operators for heuristics
- explores the middle ground between canonical heuristic and optimal cost partitioning.
- For the same set of abstractions, the post-hoc optimization heuristic dominates the canonical heuristic.
- The computation can be done in polynomial time.