Planning and Optimization
F7. Cost Partitioning

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December 8, 2025

Planning and Optimization
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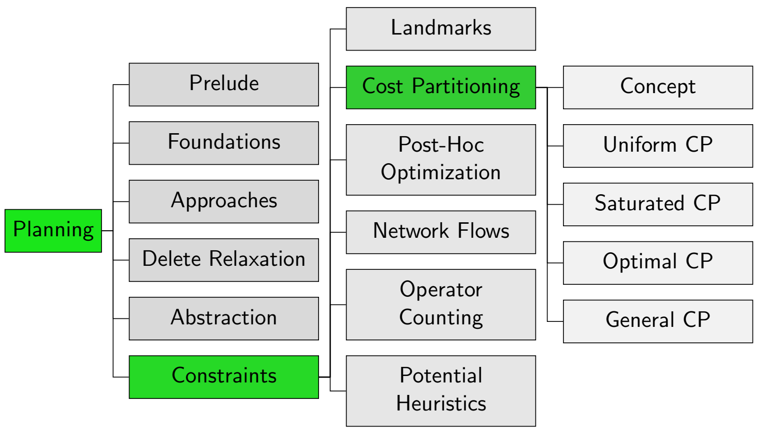
Content of the Course



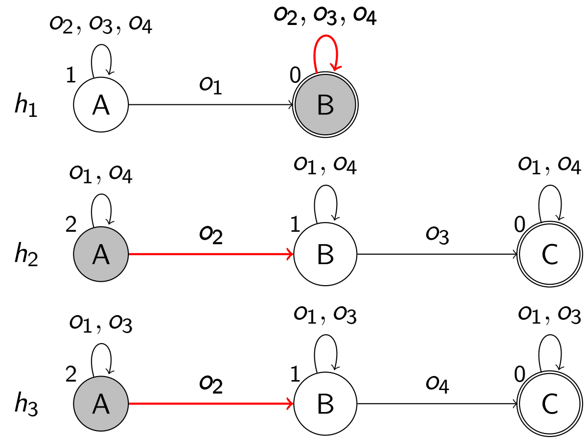
F7.1 Introduction

Exploiting Additivity

- Additivity allows to add up heuristic estimates admissibly. This gives better heuristic estimates than the maximum.
- For example, the canonical heuristic for PDBs sums up where addition is admissible (by an additivity criterion) and takes the maximum otherwise.
- Cost partitioning provides a more general additivity criterion, based on an adaption of the operator costs.

Combining Heuristics (In)admissibly: Example

Let $h = h_1 + h_2 + h_3$.



$\langle o_2, o_3, o_4 \rangle$ is a plan for $s = \langle B, A, A \rangle$ but $h(s) = 4$.
Heuristics h_2 and h_3 both account for the single application of o_2.

Solution: Cost Partitioning

The reason that h_2 and h_3 are not additive is because the cost of o_2 is considered in both.

Solution 1: We can ignore the cost of o_2 in all but one heuristic by setting its cost to 0 (e.g., $cost_3(o_2) = 0$).
This is a Zero-One cost partitioning.

Solution 2: We can equally distribute the cost of o_2 between the abstractions that use it (i.e. $cost_1(o_2) = 0$, $cost_2(o_2) = cost_3(o_2) = 0.5$).
This is a uniform cost partitioning.

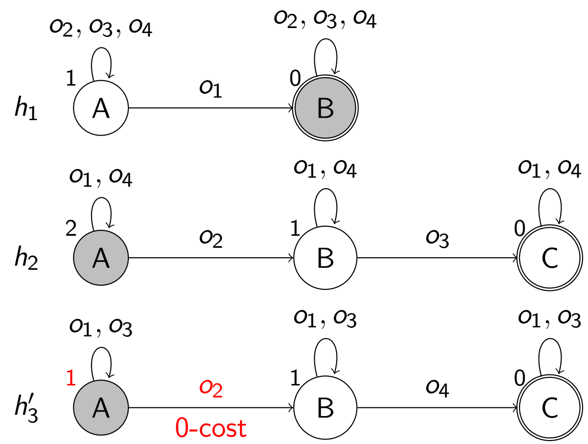
General solution: satisfy cost partitioning constraint

$$\sum_{i=1}^{n} cost_i(o) \leq cost(o) \text{ for all } o \in O$$

What about o_1, o_3 and o_4?

Combining Heuristics Admissibly: Example

Let $h' = h_1 + h_2 + h'_3$, where $h'_3 = h^{v_3}$ assuming $cost_3(o_2) = 0$.



$\langle o_2, o_3, o_4 \rangle$ is an optimal plan for $s = \langle B, A, A \rangle$ and $h'(s) = 3$ an admissible estimate.

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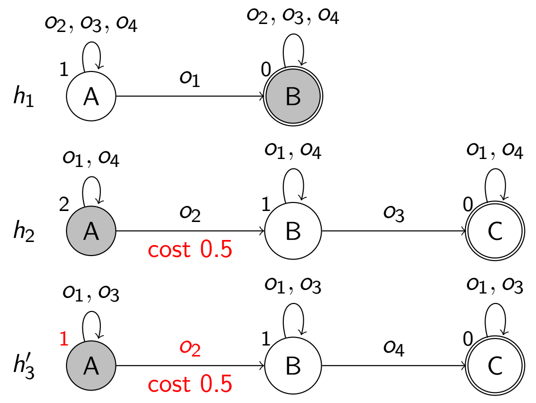
Combining Heuristics Admissibly: Example

Let $h' = h'_1 + h'_2 + h'_3$, where $h'_i = h^{v_i}$ assuming $cost_1(o_2) = 0$, $cost_2(o_2) = cost_3(o_2) = 0.5$.



$\langle o_2, o_3, o_4 \rangle$ is an optimal plan for $s = \langle B, A, A \rangle$ and $h'(s) = 0 + 1.5 + 1.5 = 3$ an admissible estimate.

Solution: Cost Partitioning

The reason that h_2 and h_3 are not additive is because the cost of o_2 is considered in both.

Solution 1: We can ignore the cost of o_2 in all but one heuristic by setting its cost to 0 (e.g., $cost_3(o_2) = 0$).
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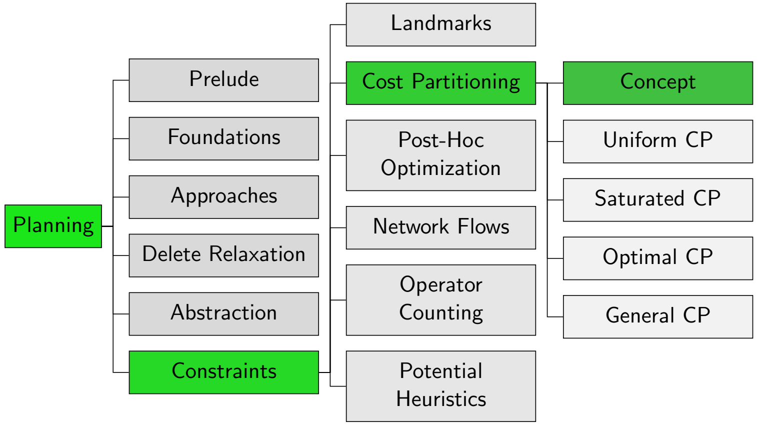
General solution: satisfy cost partitioning constraint

$$\sum_{i=1}^{n} cost_i(o) \leq cost(o) \text{ for all } o \in O$$

What about o_1, o_3 and o_4?

F7.2 Cost Partitioning

Content of the Course



Cost Partitioning

Definition (Cost Partitioning)

Let Π be a planning task with operators O.

A cost partitioning for Π is a tuple $\langle cost_1, \dots, cost_n \rangle$, where

- $cost_i : O \to \mathbb{R}_0^+$ for $1 \leq i \leq n$ and
- $\sum_{i=1}^n cost_i(o) \leq cost(o)$ for all $o \in O$.

The cost partitioning induces a tuple $\langle \Pi_1, \dots, \Pi_n \rangle$ of planning tasks, where each Π_i is identical to Π except that the cost of each operator o is $cost_i(o)$.

Cost Partitioning: Admissibility (1)

Theorem (Sum of Solution Costs is Admissible)

Let Π be a planning task, $\langle cost_1, \dots, cost_n \rangle$ be a cost partitioning and $\langle \Pi_1, \dots, \Pi_n \rangle$ be the tuple of induced tasks.

Then the sum of the solution costs of the induced tasks is an admissible heuristic for Π, i.e., $\sum_{i=1}^n h^*_{\Pi_i} \leq h^*_\Pi$.

Cost Partitioning: Admissibility (2)

Proof of Theorem.

If there is no plan for state s of Π, both sides are ∞. Otherwise, let $\pi = \langle o_1, \dots, o_m \rangle$ be an optimal plan for s. Then

$$
\begin{aligned}
\sum_{i=1}^n h^*_{\Pi_i}(s) &\leq \sum_{i=1}^n \sum_{j=1}^m cost_i(o_j) && (\pi \text{ plan in each } \Pi_i)\\
&= \sum_{j=1}^m \sum_{i=1}^n cost_i(o_j) && (\text{comm./ass. of sum})\\
&\leq \sum_{j=1}^m cost(o_j) && (\text{cost partitioning})\\
&= h^*_\Pi(s) && (\pi \text{ optimal plan in } \Pi)
\end{aligned}
$$

□

Cost Partitioning Preserves Admissibility

In the rest of the chapter, we write h_Π to denote heuristic h evaluated on task Π.

Corollary (Sum of Admissible Estimates is Admissible)

Let Π be a planning task and let $\langle \Pi_1, \dots, \Pi_n \rangle$ be induced by a cost partitioning.

For admissible heuristics $h_1, \dots, h_n$, the sum $h(s) = \sum_{i=1}^n h_{i,\Pi_i}(s)$ is an admissible estimate for s in Π.

Cost Partitioning Preserves Consistency

Theorem (Cost Partitioning Preserves Consistency)

Let Π be a planning task and let $\langle \Pi_1, \dots, \Pi_n \rangle$ be induced by a cost partitioning $\langle cost_1, \dots, cost_n \rangle$.

If $h_1, \dots, h_n$ are consistent heuristics then $h = \sum_{i=1}^n h_{i,\Pi_i}$ is a consistent heuristic for Π.

Proof.

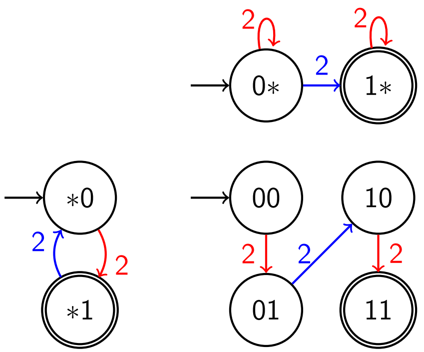
Let o be an operator that is applicable in state s.

$$\begin{aligned} h(s) &= \sum_{i=1}^{n} h_{i,\Pi_i}(s) \leq \sum_{i=1}^{n} (cost_i(o) + h_{i,\Pi_i}(s[\![o]\!])) \\ &= \sum_{i=1}^{n} cost_i(o) + \sum_{i=1}^{n} h_{i,\Pi_i}(s[\![o]\!]) \leq cost(o) + h(s[\![o]\!]) \end{aligned}$$

$\square$

Cost Partitioning: Example

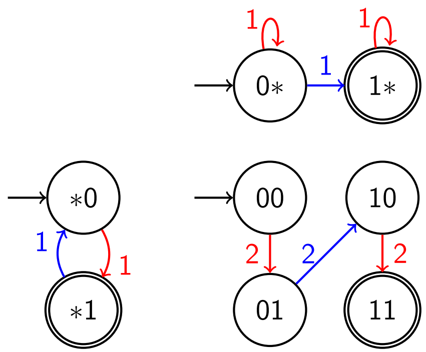
Example (No Cost Partitioning)



Heuristic value: $\max\{2, 2\} = 2$

Cost Partitioning: Example

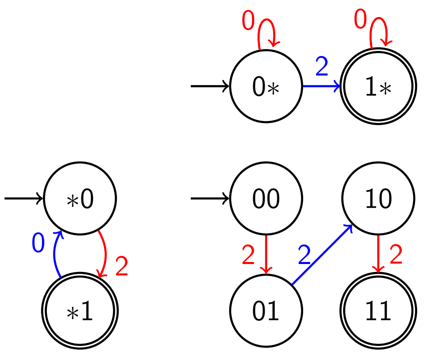
Example (Cost Partitioning 1)



Heuristic value: $1 + 1 = 2$

Cost Partitioning: Example

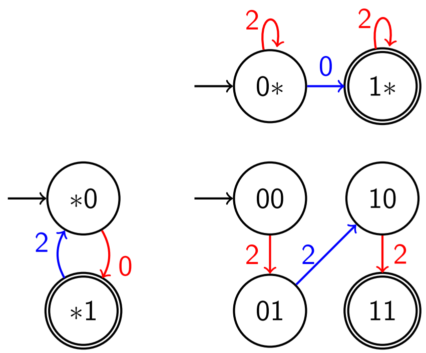
Example (Cost Partitioning 2)



Heuristic value: $2 + 2 = 4$

Cost Partitioning: Example

Example (Cost Partitioning 3)



Heuristic value: $0 + 0 = 0$

Cost Partitioning: Quality

- $h(s) = h_{1,\Pi_1}(s) + \cdots + h_{n,\Pi_n}(s)$
  can be better or worse than any $h_{i,\Pi}(s)$
  $\rightarrow$ depending on cost partitioning
- strategies for defining cost-functions
  - uniform (now)
  - zero-one
  - saturated (afterwards)
  - optimal (next chapter)

F7.3 Uniform Cost Partitioning

Content of the Course

- Planning
  - Prelude
  - Foundations
  - Approaches
  - Delete Relaxation
  - Abstraction
  - Constraints
    - Landmarks
    - Cost Partitioning
      - Concept
      - Uniform CP
      - Saturated CP
      - Optimal CP
      - General CP
    - Post-Hoc Optimization
    - Network Flows
    - Operator Counting
    - Potential Heuristics

Idea

- Principal idea: Distribute the cost of each operator equally (= uniformly) among all heuristics.
- But: Some heuristics do only account for the cost of certain operators and the cost of other operators does not affect the heuristic estimate. For example:
  - a disjunctive action landmark accounts for the contained operators,
  - a PDB heuristic accounts for all operators affecting the variables in the pattern.

$\Rightarrow$ Distribute the cost of each operator uniformly among all heuristics that account for this operator.

Example: Uniform Cost Partitioning for Landmarks

- For disjunctive action landmark L of state s in task Π', let $h_{L,\Pi'}(s)$ be the cost of L in Π'.
- Then $h_{L,\Pi'}(s)$ is admissible (in Π').
- Consider set $\mathcal{L} = \{L_1, \dots, L_n\}$ of disjunctive action landmarks for state s of task Π.
- Use cost partitioning $\langle cost_{L_1}, \dots, cost_{L_n} \rangle$, where

$$cost_{L_i}(o) = \begin{cases} cost(o)/|\{L \in \mathcal{L} \mid o \in L\}| & \text{if } o \in L_i \\ 0 & \text{otherwise} \end{cases}$$

- Let $\langle \Pi_{L_1}, \dots, \Pi_{L_n} \rangle$ be the tuple of induced tasks.
- $h(s) = \sum_{i=1}^{n} h_{L_i,\Pi_{L_i}}(s)$ is an admissible estimate for s in Π.
- h is the uniform cost partitioning heuristic for landmarks.

Example: Uniform Cost Partitioning for Landmarks

Definition (Uniform Cost Partitioning Heuristic for Landmarks)

Let $\mathcal{L}$ be a set of disjunctive action landmarks.

The uniform cost partitioning heuristic $h^{\mathrm{UCP}}(\mathcal{L})$ is defined as

$$h^{\mathrm{UCP}}(\mathcal{L}) = \sum_{L \in \mathcal{L}} \min_{o \in L} c'(o) \text{ with}$$

$$c'(o) = cost(o)/|\{L \in \mathcal{L} \mid o \in L\}|.$$

Example: Uniform Cost Partitioning for Landmarks

Example

Given disjunctive action landmarks
$L_1 = \{o_1, o_3\}, \ \ L_2 = \{o_1, o_2, o_4\}, \ \ L_3 = \{o_1, o_4, o_5\}$

with operator cost function
$c(o_1) = 6, \ \ c(o_2) = 4, \ \ c(o_3) = 1, \ \ c(o_4) = 6, \ \ c(o_5) = 3$

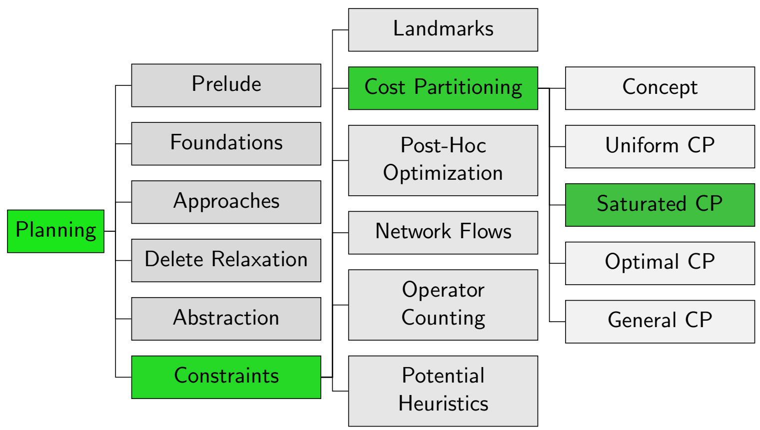
UCP for landmarks uses adapted costs
$c'(o_1) = 2, \ \ c'(o_2) = 4, \ \ c'(o_3) = 1, \ \ c'(o_4) = 3, \ \ c'(o_5) = 3$

with resulting heuristic estimate
$h^{\text{UCP}}(\{L_1, L_2, L_3\}) = 1 + 2 + 2 = 5.$

(MHS heuristic estimate: 6)

F7.4 Saturated Cost Partitioning

Content of the Course



Idea

Heuristics do not always "need" all operator costs

- Pick a heuristic and use minimum costs preserving all estimates
- Continue with remaining cost until all heuristics were picked

Saturated cost partitioning (SCP) currently offers the best tradeoff between computation time and heuristic guidance in practice.

Saturated Cost Function

Definition (Saturated Cost Function)

Let Π be a planning task and h be a heuristic.
A cost function scf is saturated for h and *cost* if

1. $\mathrm{scf}(o) \leq \mathit{cost}(o)$ for all operators o and
2. $h_{\Pi_{\mathrm{scf}}}(s) = h_{\Pi}(s)$ for all states s, where Π_{scf} is Π with cost function scf.

Minimal Saturated Cost Function

For abstractions, there exists a unique minimal saturated cost function (MSCF).

Definition (MSCF for Abstractions)

Let Π be a planning task and α be an abstraction heuristic.
The minimal saturated cost function for α is

$$\mathrm{mscf}(o) = \max\left(\max_{\alpha(s)\xrightarrow{o}\alpha(t)} h^{\alpha}(s) - h^{\alpha}(t), 0\right)$$

Algorithm

Saturated Cost Partitioning: Seipp & Helmert (2014)

Iterate:

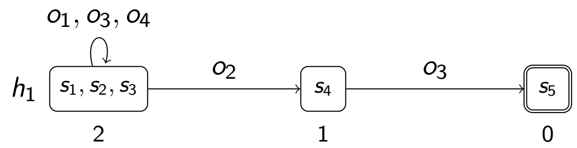
1. Pick a heuristic h_i that hasn't been picked before. Terminate if none is left.
2. Compute h_i given current *cost*
3. Compute an (ideally minimal) saturated cost function scf_i for h_i
4. Decrease $\mathit{cost}(o)$ by $\mathrm{scf}_i(o)$ for all operators o

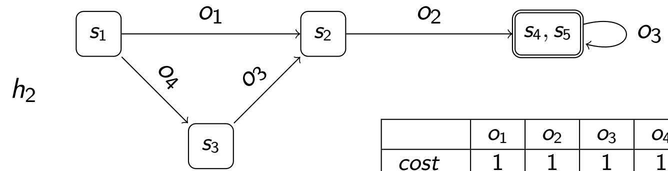
$\langle \mathrm{scf}_1, \dots, \mathrm{scf}_n \rangle$ is a saturated cost partitioning (SCP) for $\langle h_1, \dots, h_n \rangle$ (in pick order)

Example

Consider the abstraction heuristics h_1 and h_2

3. Compute minimal saturated cost function mscf_i for h_i



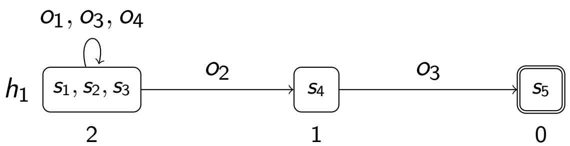


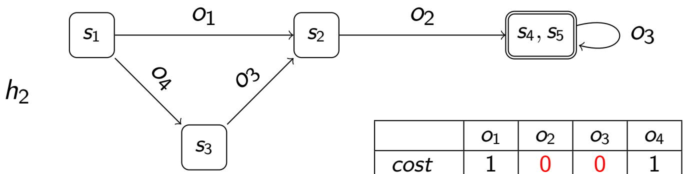
	o_1	o_2	o_3	o_4
cost	1	1	1	1
mscf_1	0	1	1	0

Example

Consider the abstraction heuristics h_1 and h_2

4. Decrease $cost(o)$ by $\mathrm{mscf}_i(o)$ for all operators o





	o_1	o_2	o_3	o_4
cost	1	0	0	1
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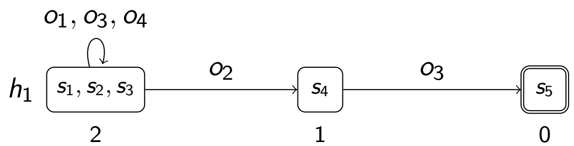
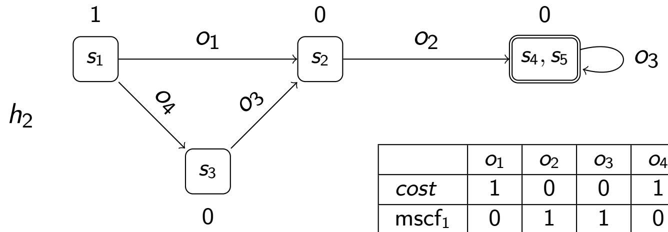
Example

Consider the abstraction heuristics h_1 and h_2

3. Compute minimal saturated cost function mscf_i for h_i



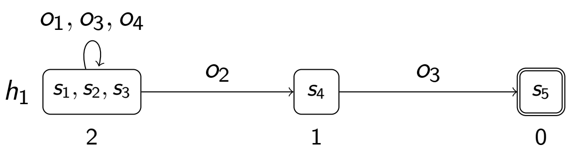


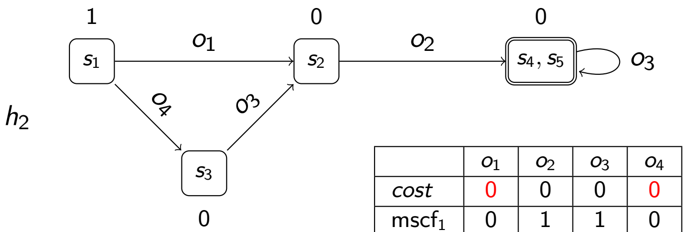
	o_1	o_2	o_3	o_4
cost	1	0	0	1
mscf_1	0	1	1	0
mscf_2	1	0	0	1

Example

Consider the abstraction heuristics h_1 and h_2

4. Decrease $cost(o)$ by $\mathrm{mscf}_i(o)$ for all operators o



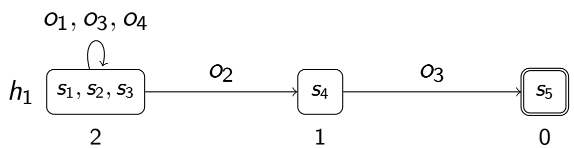


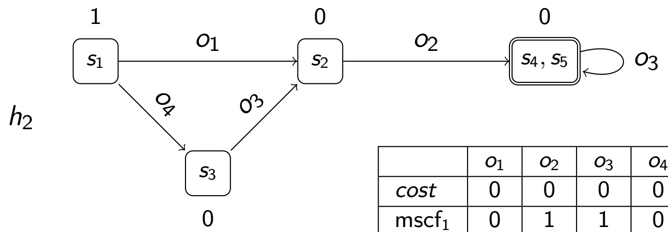
	o_1	o_2	o_3	o_4
cost	0	0	0	0
mscf_1	0	1	1	0
mscf_2	1	0	0	1

Example

Consider the abstraction heuristics h_1 and h_2

1. Pick a heuristic h_i. Terminate if none is left.





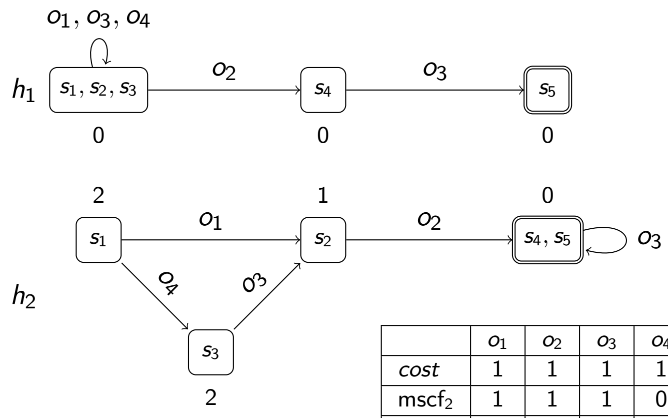
	o_1	o_2	o_3	o_4
cost	0	0	0	0
mscf_1	0	1	1	0
mscf_2	1	0	0	1

Influence of Selected Order

- quality highly susceptible to selected order
- there are almost always orders where SCP performs much better than uniform or zero-one cost partitioning
- but there are also often orders where SCP performs worse

Saturated Cost Partitioning: Order

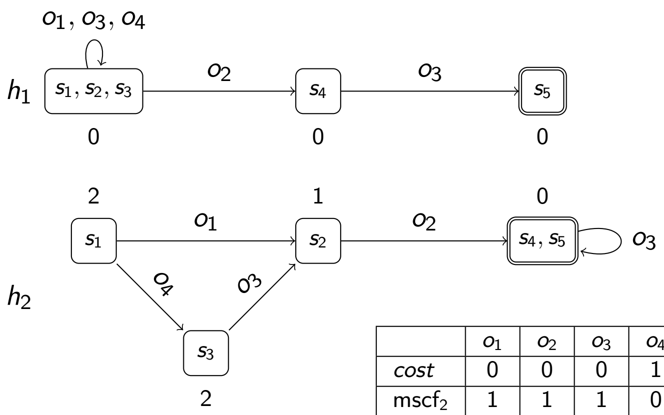
Consider the abstraction heuristics h_1 and h_2



	o_1	o_2	o_3	o_4
cost	1	1	1	1
mscf_2	1	1	1	0

Saturated Cost Partitioning: Order

Consider the abstraction heuristics h_1 and h_2



	o_1	o_2	o_3	o_4
cost	0	0	0	1
mscf_2	1	1	1	0
mscf_1	0	0	0	0

Influence of Selected Order

- quality highly susceptible to selected order
- there are almost always orders where SCP performs much better than uniform or zero-one cost partitioning
- but there are also often orders where SCP performs worse

Maximizing over multiple orders good solution in practice

SCP for Disjunctive Action Landmarks

For disjunctive action landmarks we also know how to compute a minimal saturated cost function:

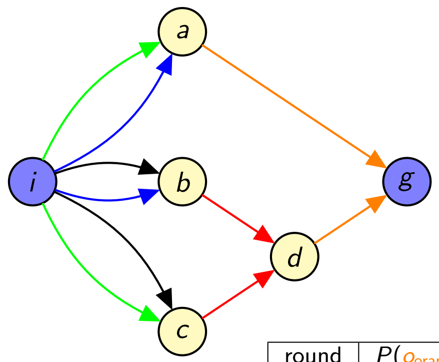
Definition (MSCF for Disjunctive Action Landmark)

Let Π be a planning task and $\mathcal{L}$ be a disjunctive action landmark. The minimal saturated cost function for $\mathcal{L}$ is

$$\mathrm{mscf}(o) = \begin{cases} \min_{o \in \mathcal{L}} cost(o) & \text{if } o \in \mathcal{L} \\ 0 & \text{otherwise} \end{cases}$$

Does this look familiar?

Reminder: LM-Cut



$o_{\text{blue}} = \langle \{i\}, \{a, b\}, \{\}, 4 \rangle$
$o_{\text{green}} = \langle \{i\}, \{a, c\}, \{\}, 5 \rangle$
$o_{\text{black}} = \langle \{i\}, \{b, c\}, \{\}, 3 \rangle$
$o_{\text{red}} = \langle \{b, c\}, \{d\}, \{\}, 2 \rangle$
$o_{\text{orange}} = \langle \{a, d\}, \{g\}, \{\}, 0 \rangle$

round	$P(o_{\text{orange}})$	$P(o_{\text{red}})$	landmark	cost
1	d	b	$\{o_{\text{red}}\}$	2
2	a	b	$\{o_{\text{green}}, o_{\text{blue}}\}$	4
3	d	c	$\{o_{\text{green}}, o_{\text{black}}\}$	1
			$h^{\text{LM-cut}}(I)$	7

SCP for Disjunctive Action Landmarks

Same algorithm can be used for disjunctive action landmarks, where we also have a minimal saturated cost function.

Definition (MSCF for Disjunctive Action Landmark)

Let Π be a planning task and $\mathcal{L}$ be a disjunctive action landmark. The minimal saturated cost function for $\mathcal{L}$ is

$$\mathrm{mscf}(o) = \begin{cases} \min_{o \in \mathcal{L}} cost(o) & \text{if } o \in \mathcal{L} \\ 0 & \text{otherwise} \end{cases}$$

Does this look familiar?

LM-Cut computes SCP over disjunctive action landmarks

F7.5 Summary

Summary

- Cost partitioning allows to admissibly add up estimates of several heuristics.
- This can be better or worse than the best individual heuristic on the original problem, depending on the cost partitioning.
- Uniform cost partitioning distributes the cost of each operator uniformly among all heuristics that account for it.
- Saturated cost partitioning offers a good tradeoff between computation time and heuristic guidance.
- LM-Cut computes a SCP over disjunctive action landmarks.