Planning and Optimization

E13. Cartesian Abstractions

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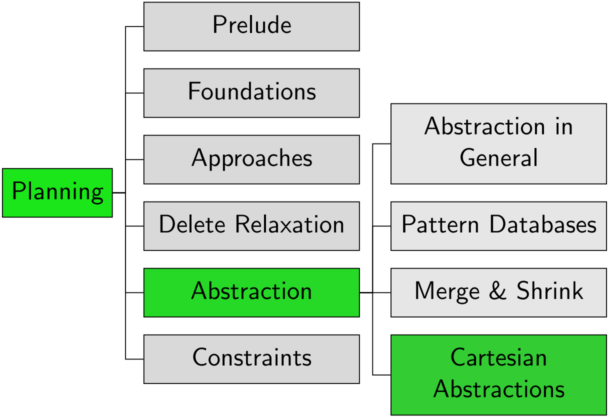
Content of the Course



Introduction

Counterexample-Guided Abstraction Refinement

Counterexample-guided abstraction refinement (CEGAR) is an approach to compute a tailored abstraction for a task (or to solve it).

- Start with a very coarse abstraction.
- Iteratively compute an (optimal) abstract solution and check whether it works for the concrete tasks.
  - If yes, the task is solved.
  - If not, refine the abstraction so that the same flaw will not be encountered in future iterations.

CEGAR is another technique originally introduced for model checking.

Our Plan for Today

- For a certain class of abstractions (the Cartesian abstractions), CEGAR can be efficiently implemented.
- In this chapter, we get to know this class of abstractions and the necessary foundations.
- In the next chapter, we see how they can be used within CEGAR.

Remarks

- In Ch. E13 and E14 we continue to only consider SAS^+ tasks.
- To facilitate notation, we will use an arbitrary (but fixed) order on the variables.
  $\rightarrow$ Tuple of variables instead of set of variables.
- These chapters are based on:
  Jendrik Seipp and Malte Helmert.
  Counterexample-Guided Cartesian Abstraction Refinement for Classical Planning. Journal of Artificial Intelligence Research 62, pp. 535-577. 2018.

Example Task: Two Packages, One Truck

In E13 and E14 we use the following running example.

Example (Two Packages, One Truck)

Consider the following FDR planning task $\langle V, I, O, \gamma \rangle$:

- $V = \{p_\mathsf{A}, p_\mathsf{B}, t\}$ with
  - $\mathrm{dom}(p)_\mathsf{A} = \mathrm{dom}(p_\mathsf{B}) = \{\mathsf{L}, \mathsf{I}, \mathsf{R}\}$
  - $\mathrm{dom}(t) = \{\mathsf{L}, \mathsf{R}\}$
- $I = \{p_\mathsf{A} \mapsto \mathsf{L}, p_\mathsf{B} \mapsto \mathsf{L}, t \mapsto \mathsf{L}\}$
- $O = \{\mathrm{pickup}_{i,j} \mid i \in \{\mathsf{A}, \mathsf{B}\}, j \in \{\mathsf{L}, \mathsf{R}\}\}$
  $\cup \{\mathrm{drop}_{i,j} \mid i \in \{\mathsf{A}, \mathsf{B}\}, j \in \{\mathsf{L}, \mathsf{R}\}\}$
  $\cup \{\mathrm{move}_{i,j} \mid i, j \in \{\mathsf{L}, \mathsf{R}\}, i \neq j\}$, where
  - $\mathrm{pickup}_{i,j} = \langle p_i = j \land t = j, p_i := \mathsf{I}, 1 \rangle$
  - $\mathrm{drop}_{i,j} = \langle p_i = \mathsf{I} \land t = j, p_i := j, 1 \rangle$
  - $\mathrm{move}_{i,j} = \langle t = i, t := j, 1 \rangle$
- $\gamma = (p_\mathsf{A} = \mathsf{R} \land p_\mathsf{B} = \mathsf{R})$

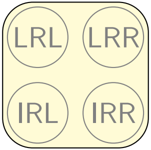
Cartesian Sets

Cartesian Sets

Definition

A set of states for a planning task with variables $\langle v_1, \dots, v_n \rangle$ is called Cartesian if it is of the form $A_1 \times \dots \times A_n$, where $A_i \subseteq \mathrm{dom}(v_i)$ for all $1 \le i \le n$.

$\{L, I\} \times \{R\} \times \{L, R\} = \{(L, R, L), (L, R, R), (I, R, L), (I, R, R)\}$
for variables $\langle p_A, p_B, t \rangle$



Conjunctions of Atoms as Cartesian Sets

For a conjunction φ of atoms, the set of all states s with $s \models \varphi$ is Cartesian and can be defined as follows:

Definition

Let φ be a conjunction of atoms over finite domain variables $V = \langle v_1, \dots, v_n \rangle$. The Cartesian set induced by φ is $\mathit{Cartesian}(\varphi) = A_1 \times \dots \times A_n$, where

$$A_i = \begin{cases} \mathrm{dom}(v_i) & \text{if } \varphi \text{ contains no atom } v_i = d, \\ \{d\} & \text{if } \varphi \text{ contains an atom } v_i = d \text{ and} \\ & \quad \text{no atom } v_i = d' \text{ with } d \neq d' \\ \emptyset & \text{otherwise (conflicting atoms for } v_i\text{).} \end{cases}$$

Conjunctions of Atoms as Cartesian Sets: Examples

In the running example with variables $\langle p_A, p_B, t\rangle$

- $Cartesian(p_A = R \wedge t = L) = \{R\} \times \{L, I, R\} \times \{L\}$
- $Cartesian(p_A = R \wedge t = L \wedge t = R) = \{R\} \times \{L, I, R\} \times \emptyset$

Properties of Cartesian Sets

Theorem

Let $\Pi = \langle V, O, I, \gamma \rangle$ *be a* SAS^+ *planning task.*

1. *The set of goal states of* Π *is Cartesian.*
2. *For all* $o \in O$*, the set of states in which* o *is applicable is Cartesian.*
3. *The intersection of Cartesian sets over the same variables is Cartesian.*
4. *For all operators* o*, the regression of a Cartesian set wrt.* o *is Cartesian.*

From the proofs we will see that the corresponding Cartesian sets are easy to determine.

Properties of Cartesian Sets

Proof Sketch.

1. The set of goal states is *Cartesian*(γ).

...

Properties of Cartesian Sets

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...

Properties of Cartesian Sets

Proof Sketch.

1. The set of goal states is $\mathit{Cartesian}(\gamma)$.
2. For $o \in O$, the set of states in which o is applicable is $\mathit{Cartesian}(\mathit{pre}(o))$.
3. The intersection of Cartesian sets $A_1 \times \cdots \times A_n$ and $B_1 \times \cdots \times B_n$ is $(A_1 \cap B_1) \times \cdots \times (A_n \cap B_n)$.

...

Properties of Cartesian Sets

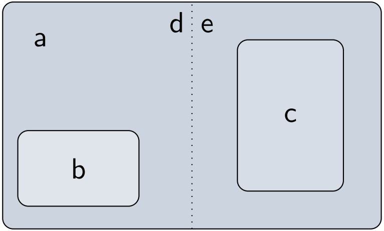
Proof Sketch (continued).

4. With variables $\langle v_1, \ldots, v_n \rangle$, the regression of Cartesian set $b = B_1 \times \cdots \times B_n$ wrt. o is $regr(b, o) = A_1 \times \cdots \times A_n$, where

$$A_i = \begin{cases} B_i & \text{if } v_i \text{ does not occur in } pre(o) \text{ and } eff(o) \\ \emptyset & \text{if } o \text{ has an effect setting } v_i \text{ to } d' \notin B_i \\ & \text{or if } o \text{ has no effect on } v_i \\ & \quad \text{but a precondition } v_i = d \text{ with } d \notin B_i. \\ \text{dom}(v_i) & \text{if } o \text{ has no precondition on } v_i \text{ and} \\ & \quad \text{an effect setting } v_i \text{ to } d' \in B_i \\ \{d\} & \text{if } o \text{ has a precondition } v_i = d \text{ and} \\ & \quad \text{an effect setting } v_i \text{ to } d' \in B_i \\ & \text{or if } o \text{ has precondition } v_i = d \text{ with } d \in B_i \\ & \quad \text{and no effect on } v_i \end{cases}$$

□

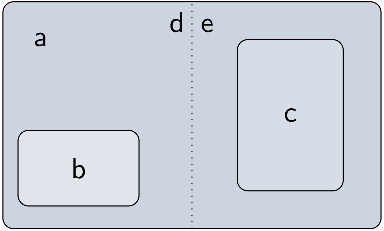
Splitting Cartesian Sets

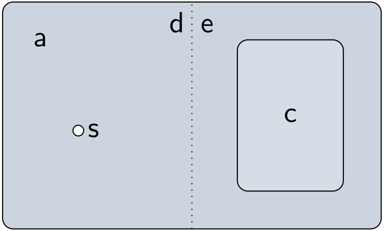


Theorem (Splits)

1. *If $b \subseteq a$ and $c \subseteq a$ are disjoint Cartesian subsets of the Cartesian set a, then a can be partitioned into Cartesian sets d and e with $b \subseteq d$ and $c \subseteq e$.*

Splitting Cartesian Sets

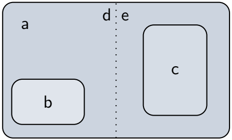




Theorem (Splits)

1. *If $b \subseteq a$ and $c \subseteq a$ are disjoint Cartesian subsets of the Cartesian set a, then a can be partitioned into Cartesian sets d and e with $b \subseteq d$ and $c \subseteq e$.*
2. *If $c \subseteq a$ is a Cartesian subset of the Cartesian set a and $s \in a \setminus c$, then a can be partitioned into Cartesian sets d and e with $s \in d$ and $c \subseteq e$.*

Splitting Cartesian Sets



Proof.

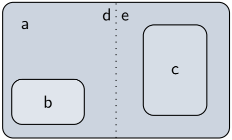
For 1), let $a = A_1 \times \cdots \times A_n$, $b = B_1 \times \cdots \times B_n$ and $c = C_1 \times \cdots \times C_n$.

Let j be such that B_j and C_j are disjoint. It must exist because otherwise b and c are not disjoint (we could select for each variable v_i a value in $B_i \cap C_i$).

Partition A_j into D_j and E_j with $B_j \subseteq D_j$ and $C_j \subseteq E_j$, e.g. $E_j = C_j$ and $D_j = A_j \setminus C_j$.

Then $d = A_1 \times \cdots \times A_{j-1} \times D_j \times A_{j+1} \times \cdots \times A_n$ and $e = A_1 \times \cdots \times A_{j-1} \times E_j \times A_{j+1} \times \cdots \times A_n$

Splitting Cartesian Sets



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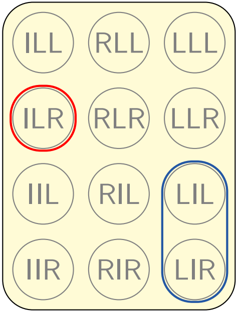
2) follows from 1) by setting $b = \{s\}$ (a Cartesian set). □

Splitting Cartesian Sets: Example

$$a : \{I, R, L\} \times \{L, I\} \times \{L, R\}$$

$$b : \{I\} \times \{L\} \times \{R\}$$



$$c : \{L\} \times \{I\} \times \{L, R\}$$

On which variable(s) can we split?

Splitting Cartesian Sets: Example

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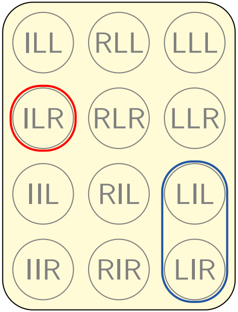
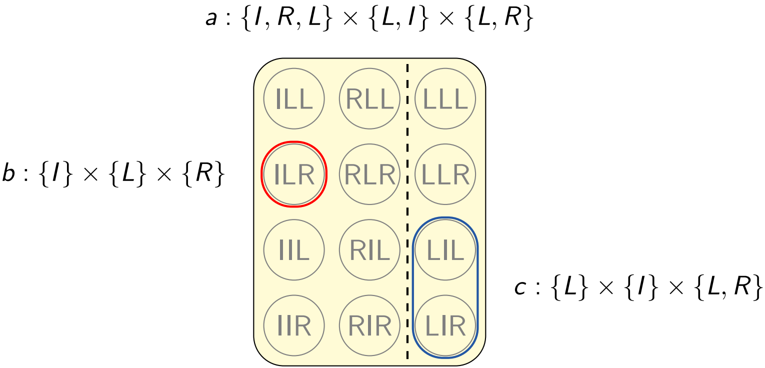


$c : \{L\} \times \{I\} \times \{L, R\}$

On which variable(s) can we split? $\rightsquigarrow$ first or second.
What are the two Cartesian sets d and e in each case?

Splitting Cartesian Sets: Example

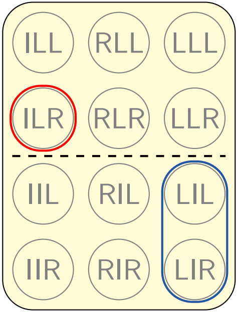


Split on first variable:
$d = \{I, R\} \times \{L, I\} \times \{L, R\}$ and $e = \{L\} \times \{L, I\} \times \{L, R\}$

Splitting Cartesian Sets: Example

$a : \{I, R, L\} \times \{L, I\} \times \{L, R\}$

$b : \{I\} \times \{L\} \times \{R\}$



$c : \{L\} \times \{I\} \times \{L, R\}$

Split on second variable:
$d = \{I, R, L\} \times \{L\} \times \{L, R\}$ and $e = \{I, R, L\} \times \{I\} \times \{L, R\}$

Cartesian Abstractions

Reminder: Abstractions as Equivalence Relations

- An abstraction α induces the equivalence relation $\sim_\alpha$ over the set of (concrete) states as $s \sim_\alpha t$ iff $\alpha(s) = \alpha(t)$.

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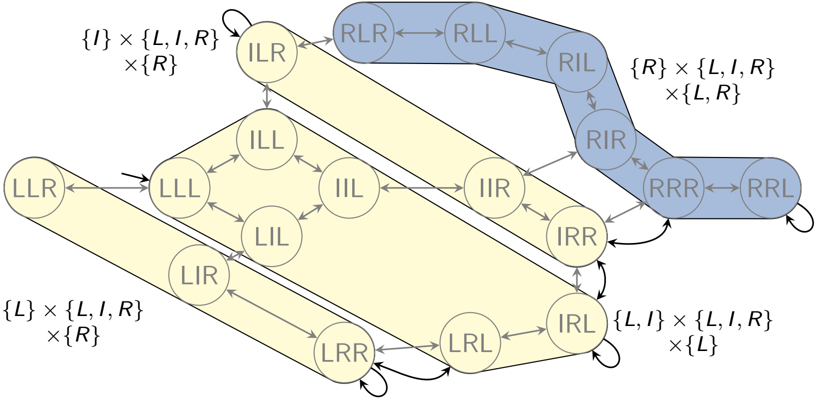
- An abstraction α induces the equivalence relation $\sim_\alpha$ over the set of (concrete) states as $s \sim_\alpha t$ iff $\alpha(s) = \alpha(t)$.
- The equivalence class $[s]_\alpha$ of state s is the set of all concrete states that are mapped to the same abstract state as s.
- We write $\sim$ and $[s]$, if α is clear from context.

Cartesian Abstraction

Definition

An abstraction α is called Cartesian if all equivalence classes of $\sim_\alpha$ are Cartesian sets.

Example



Labels omitted for clarity.

Relationship to other Classes of Abstractions

- Cartesian abstractions generalize projections (PDBs): the equivalence classes of projections are Cartesian.
- Merge & Shrink abstractions are more general than Cartesian abstractions (every abstraction can be represented as Merge & Shrink abstraction).
- Merge & Shrink and Cartesian abstractions are incomparable in representation size: there are compact Cartesian abstractions that do not have a compact Merge & Shrink representation and vice versa.

Summary

Summary

- Cartesian sets are sets of states that can be represented as a Cartesian product of possible values for each variable.
- In Cartesian abstractions the sets of states that do not get distinguished must be Cartesian.