Planning and Optimization

E11. Merge-and-Shrink: Properties and Shrink Strategies

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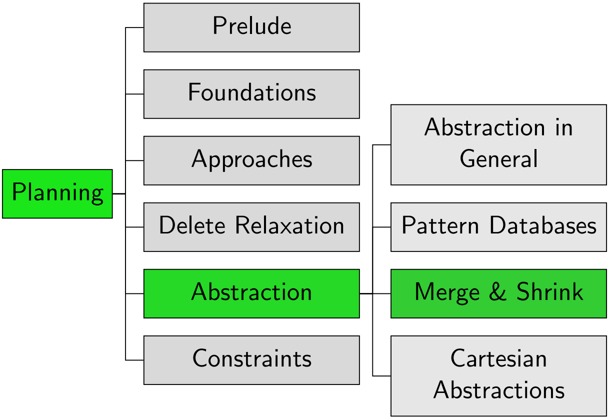
Content of the Course



Reminder: Generic Algorithm Template

Generic Merge & Shrink Algorithm for planning task Π

$F := F(\Pi)$
while $|F| > 1$:
  select *type* $\in \{\text{merge}, \text{shrink}\}$
  if *type* $=$ merge:
    select $\mathcal{T}_1, \mathcal{T}_2 \in F$
    $F := (F \setminus \{\mathcal{T}_1, \mathcal{T}_2\}) \cup \{\mathcal{T}_1 \otimes \mathcal{T}_2\}$
  if *type* $=$ shrink:
    select $\mathcal{T} \in F$
    choose an abstraction mapping β on $\mathcal{T}$
    $F := (F \setminus \{\mathcal{T}\}) \cup \{\mathcal{T}^{\beta}\}$
return the remaining factor $\mathcal{T}^{\alpha}$ in F

Heuristic Properties

Properties of Merge-and-Shrink Heuristics

To understand merge-and-shrink abstractions better,
we are interested in the properties of the resulting heuristic:

- Is it admissible ($h^\alpha(s) \leq h^*(s)$ for all states s)?
- Is it consistent ($h^\alpha(s) \leq c(o) + h^\alpha(t)$ for all trans. $s \xrightarrow{o} t$)?
- Is it perfect ($h^\alpha(s) = h^*(s)$ for all states s)?

Because merge-and-shrink is a generic procedure,
the answers may depend on how exactly we instantiate it:

- size limits
- merge strategy
- shrink strategy

Merge-and-Shrink as Sequence of Transformations

- Consider a run of the merge-and-shrink construction algorithm with n iterations of the main loop.
- Let F_i $(0 \leq i \leq n)$ be the FTS F after i loop iterations.
- Let $\mathcal{T}_i$ $(0 \leq i \leq n)$ be the transition system represented by F_i, i.e., $\mathcal{T}_i = \bigotimes F_i$.
- In particular, $F_0 = F(\Pi)$ and $F_n = \{\mathcal{T}_n\}$.
- For SAS$^+$ tasks Π, we also know $\mathcal{T}_0 = \mathcal{T}(\Pi)$.

For a formal study, it is useful to view merge-and-shrink construction as a sequence of transformations from $\mathcal{T}_i$ to $\mathcal{T}_{i+1}$.

(We do it in a bit more general fashion than necessary for merge and shrink steps only, to also cover some improvements we will see later.)

Transformations

Definition (Transformation)

Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$ and $\mathcal{T}' = \langle S', L', c', T', s'_0, S'_\star \rangle$ be transition systems.
Let $\sigma : S \to S'$ map the states of $\mathcal{T}$ to the states of $\mathcal{T}'$ and $\lambda : L \to L'$ map the labels of $\mathcal{T}$ to the labels of $\mathcal{T}'$.

The tuple $\tau = \langle \mathcal{T}, \sigma, \lambda, \mathcal{T}' \rangle$ is called a transformation from $\mathcal{T}$ to $\mathcal{T}'$. We also write it as $\mathcal{T} \xrightarrow{\sigma,\lambda} \mathcal{T}'$.

The transformation τ induces the heuristic h^τ for $\mathcal{T}$ defined as $h^\tau(s) = h^*_{\mathcal{T}'}(\sigma(s))$.

Example: If α is an abstraction mapping for transition system $\mathcal{T}$, then $\mathcal{T} \xrightarrow{\alpha,\mathrm{id}} \mathcal{T}^\alpha$ is a transformation.

Conservative Transformations

Definition (Conservative Transformation)

Let $\mathcal{T}$ and $\mathcal{T}'$ be transition systems with label sets L and L' and cost functions c and c', respectively.

A transformation $\langle \mathcal{T}, \sigma, \lambda, \mathcal{T}' \rangle$ is conservative if

- $c'(\lambda(\ell)) \leq c(\ell)$ for all $\ell \in L$,
- for all transitions $\langle s, \ell, t \rangle$ of $\mathcal{T}$ there is a transition $\langle \sigma(s), \lambda(\ell), \sigma(t) \rangle$ of $\mathcal{T}'$, and
- for all goal states s of $\mathcal{T}$, state $\sigma(s)$ is a goal state of $\mathcal{T}'$.

Example: If α is an abstraction mapping for transition system $\mathcal{T}$, then $\mathcal{T} \xrightarrow{\alpha,\mathrm{id}} \mathcal{T}^\alpha$ is a conservative transformation.

Conservative Transformations: Heuristic Properties (1)

Theorem

If τ is a conservative transformation from transition system $\mathcal{T}$ to transition system $\mathcal{T}'$ then h^τ is a safe, consistent, goal-aware and admissible heuristic for $\mathcal{T}$.

Proof.

We prove goal-awareness and consistency, the other properties follow from these two.

Goal-awareness: For all goal states $s_\star$ of $\mathcal{T}$, state $\sigma(s_\star)$ is a goal state of $\mathcal{T}'$ and therefore $h^\tau(s_\star) = h^*_{\mathcal{T}'}(\sigma(s_\star)) = 0$. ...

Conservative Transformations: Heuristic Properties (2)

Proof (continued).

Consistency: Let c and c' be the label cost functions of $\mathcal{T}$ and $\mathcal{T}'$, respectively. Consider state s of $\mathcal{T}$ and transition $\langle s, \ell, t\rangle$.
As $\mathcal{T}'$ has a transition $\langle \sigma(s), \lambda(\ell), \sigma(t)\rangle$, it holds that

$$\begin{aligned} h^{\tau}(s) &= h^*_{\mathcal{T}'}(\sigma(s)) \\ &\leq c'(\lambda(\ell)) + h^*_{\mathcal{T}'}(\sigma(t)) \\ &= c'(\lambda(\ell)) + h^{\tau}(t) \\ &\leq c(\ell) + h^{\tau}(t) \end{aligned}$$

The second inequality holds due to the requirement on the label costs. $\square$

Exact Transformations

Definition (Exact Transformation)

Let $\mathcal{T}$ and $\mathcal{T}'$ be transition systems with label sets L and L' and cost functions c and c', respectively.

A transformation $\langle \mathcal{T}, \sigma, \lambda, \mathcal{T}' \rangle$ is exact if it is conservative and

1. if $\langle s', \ell', t' \rangle$ is a transition of $\mathcal{T}'$ then for all $s \in \sigma^{-1}(s')$ there is a transition $\langle s, \ell, t \rangle$ of $\mathcal{T}$ with $t \in \sigma^{-1}(t')$ and $\ell \in \lambda^{-1}(\ell')$,
2. if s' is a goal state of $\mathcal{T}'$ then all states $s \in \sigma^{-1}(s')$ are goal states of $\mathcal{T}$, and
3. $c(\ell) = c'(\lambda(\ell))$ for all $\ell \in L$.

$\rightsquigarrow$ no "new" transitions and goal states, no cheaper labels

Heuristic Properties with Exact Transformations (1)

Theorem

If τ is an exact transformation from transition system $\mathcal{T}$ to transition system $\mathcal{T}'$ then h^τ is the perfect heuristic h^* for $\mathcal{T}$.

Proof.

As the transformation is conservative, h^τ is admissible for $\mathcal{T}$ and therefore $h^*_{\mathcal{T}}(s) \geq h^\tau(s)$.
For the other direction, we show that for every state s' of $\mathcal{T}'$ and goal path π' for s', there is for each $s \in \sigma^{-1}(s')$ a goal path in $\mathcal{T}$ that has the same cost. ...

Heuristic Properties with Exact Transformations (2)

Proof (continued).

Proof via induction over the length of π'.

$|\pi'| = 0$: If s' is a goal state of $\mathcal{T}'$ then each $s \in \sigma^{-1}(s')$ is a goal state of $\mathcal{T}$ and the empty path is a goal path for s in $\mathcal{T}$.

$|\pi'| = i + 1$: Let $\pi' = \langle s', \ell', t' \rangle \pi'_{t'}$, where $\pi'_{t'}$ is a goal path of length i from t'. Then there is for each $t \in \sigma^{-1}(t')$ a goal path π_t of the same cost in $\mathcal{T}$ (by ind. hypothesis). Furthermore, for all $s \in \sigma^{-1}(s')$ there is a state $t \in \sigma^{-1}(t')$ and a label $\ell \in \lambda^{-1}(\ell')$ such that $\mathcal{T}$ has a transition $\langle s, \ell, t \rangle$. The path $\pi = \langle s, \ell, t \rangle \pi_t$ is a solution for s in $\mathcal{T}$. As ℓ and ℓ' must have the same cost and π_t and $\pi'_{t'}$ have the same cost, π has the same cost as π'. $\square$

Composing Transformations

Merge-and-shrink performs many transformations in sequence.
We can formalize this with a notion of composition:

- Given $\tau = \mathcal{T} \xrightarrow{\sigma, \lambda} \mathcal{T}'$ and $\tau' = \mathcal{T}' \xrightarrow{\sigma', \lambda'} \mathcal{T}''$, their composition $\tau'' = \tau' \circ \tau$ is defined as $\tau'' = \mathcal{T} \xrightarrow{\sigma' \circ \sigma, \lambda' \circ \lambda} \mathcal{T}''$.
- If τ and τ' are conservative, then $\tau' \circ \tau$ is conservative.
- If τ and τ' are exact, then $\tau' \circ \tau$ is exact.

Merge-and-Shrink Transformations

F: factored transition system

Replacement with Synchronized Product is Conservative and Exact

Let $\mathcal{T}_1, \mathcal{T}_2 \in F$ with $\mathcal{T}_1 \neq \mathcal{T}_2$.
Let $F' := (X \setminus \{\mathcal{T}_1, \mathcal{T}_2\}) \cup \{\mathcal{T}_1 \otimes \mathcal{T}_2\}$.
Then there is an exact transformation $\langle \otimes F, \sigma, \mathrm{id}, \otimes F' \rangle$.

Up to the isomorphism we know from the synchronized product, we can use $\sigma = \mathrm{id}$.

Abstraction is Conservative

Let α be an abstraction of $\mathcal{T}_i \in F$ and let $F' := (F \setminus \{\mathcal{T}_i\}) \cup \{\mathcal{T}_i^\alpha\}$.
The transformation $\langle \otimes F, \sigma, \mathrm{id}, \otimes F' \rangle$ with
$\sigma(\langle s_1, \dots, s_n \rangle) = \langle s_1, \dots, s_{i-1}, \alpha(s_i), s_{i+1}, \dots, s_n \rangle$ is conservative.

(Proofs omitted.)

Properties of Merge-and-Shrink Heuristics

We can conclude the following properties
of merge-and-shrink heuristics for SAS^+ tasks:

- The heuristic is always admissible and consistent
(because it is induced by a a composition of conservative transformations).
- If all shrink transformation used are exact,
the heuristic is perfect (because it is induced by
a composition of exact transformations).

Shrink Strategies

Reminder: Generic Algorithm Template

$F := F(\Pi)$
while $|F| > 1$:
  select $\mathit{type} \in \{\text{merge}, \text{shrink}\}$
  if $\mathit{type} = \text{merge}$:
    select $\mathcal{T}_1, \mathcal{T}_2 \in F$
    $F := (F \setminus \{\mathcal{T}_1, \mathcal{T}_2\}) \cup \{\mathcal{T}_1 \otimes \mathcal{T}_2\}$
  if $\mathit{type} = \text{shrink}$:
    select $\mathcal{T} \in F$
    choose an abstraction mapping β on $\mathcal{T}$
    $F := (F \setminus \{\mathcal{T}\}) \cup \{\mathcal{T}^{\beta}\}$
return the remaining factor $\mathcal{T}^{\alpha}$ in F

Remaining Questions:

- Which abstractions to select for merging? $\rightsquigarrow$ merge strategy
- How to shrink an abstraction? $\rightsquigarrow$ shrink strategy

Shrink Strategies

How to shrink an abstraction?

We cover two common approaches:

- f-preserving shrinking
- bisimulation-based shrinking

f-preserving Shrink Strategy

f-preserving Shrink Strategy

Repeatedly combine abstract states with
identical abstract goal distances (h values) and
identical abstract initial state distances (g values).

Rationale: preserves heuristic value and overall graph shape

Tie-breaking Criterion

Prefer combining states where $g + h$ is high.
In case of ties, combine states where h is high.

Rationale: states with high $g + h$ values are less likely to be explored by A*, so inaccuracies there matter less

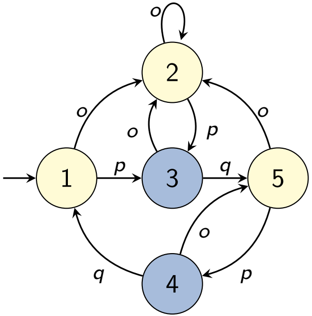
Bisimulation

Definition (Bisimulation)

Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$ be a transition system. An equivalence relation $\sim$ on S is a bisimulation for $\mathcal{T}$ if for every $\langle s, \ell, s' \rangle \in T$ and every $t \sim s$ there is a transition $\langle t, \ell, t' \rangle \in T$ with $t' \sim s'$.

A bisimulation $\sim$ is goal-respecting if $s \sim t$ implies that either $s, t \in S_\star$ or $s, t \notin S_\star$.

Bisimulation: Example



$\sim$ with equivalence classes $\{\{1,2,5\},\{3,4\}\}$ is a goal-respecting bisimulation.

Bisimulation Abstractions

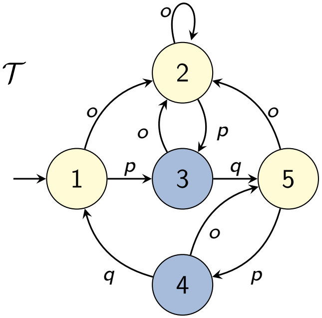
Definition (Abstractions as Bisimulation)

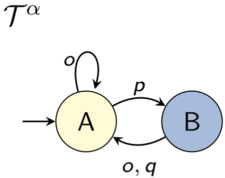
Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$ be a transition system and $\alpha : S \to S'$ be an abstraction of $\mathcal{T}$. The abstraction induces the equivalence relation $\sim_\alpha$ as $s \sim_\alpha t$ iff $\alpha(s) = \alpha(t)$.

We say that α is a (goal-respecting) bisimulation for $\mathcal{T}$ if $\sim_\alpha$ is a (goal-respecting) bisimulation for $\mathcal{T}$.

Abstraction as Bisimulations: Example

Abstraction α with
$\alpha(1) = \alpha(2) = \alpha(5) = A$ and $\alpha(3) = \alpha(4) = B$
is a goal-respecting bisimulation for $\mathcal{T}$.





Goal-respecting Bisimulations are Exact

Theorem

Let F be a factored transition system and α be an abstraction of $\mathcal{T}_i \in F$.
If α is a goal-respecting bisimulation then the transformation $\langle \otimes F, \sigma, id, \otimes F' \rangle$ with

- $\sigma(\langle s_1, \ldots, s_n \rangle) = \langle s_1, \ldots, s_{i-1}, \alpha(s_i), s_{i+1}, \ldots, s_n \rangle$ *and*
- $F' := (F \setminus \{\mathcal{T}_i\}) \cup \{\mathcal{T}_i^\alpha\}$

is exact.

(Proofs omitted.)

Shrinking with bisimulation preserves the heuristic estimates.

Bisimulations: Discussion

- As all bisimulations preserve all relevant information, we are interested in the coarsest such abstraction (to shrink as much as possible).
- There is always a unique coarsest bisimulation for $\mathcal{T}$ and it can be computed efficiently (from the explicit representation).
- In some cases, computing the bisimulation is still too expensive or it cannot sufficiently shrink a transition system.

Summary

Summary

- Merge-and-shrink abstractions can be analyzed by viewing them as a sequence of transformations.
- We only use conservative transformations, and hence merge-and-shrink heuristics for SAS^+ tasks are admissible and consistent.
- Merge-and-shrink heuristics for SAS^+ tasks that only use exact transformations are perfect.
- Bisimulation is an exact shrinking method.