Planning and Optimization

E9. Merge-and-Shrink: Factored Transition Systems

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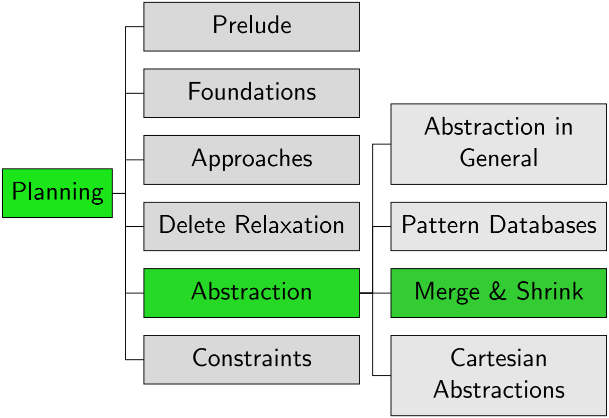
Content of the Course

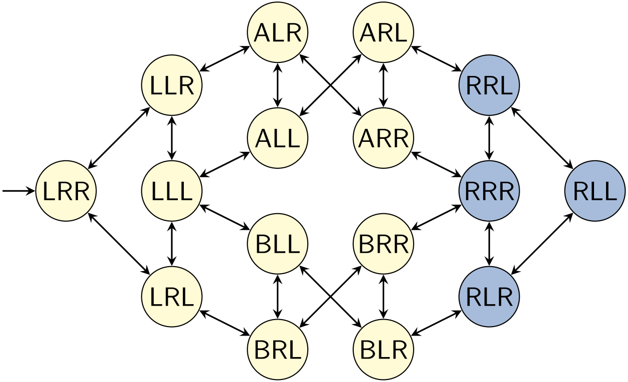


E9.1 Motivation

Beyond Pattern Databases

- Despite their popularity, pattern databases have some fundamental limitations ($\rightsquigarrow$ example on next slides).
- Today and next time, we study a class of abstractions called merge-and-shrink abstractions.
- Merge-and-shrink abstractions can be seen as a proper generalization of pattern databases.
  - They can do everything that pattern databases can do (modulo polynomial extra effort).
  - They can do some things that pattern databases cannot.

Back to the Running Example

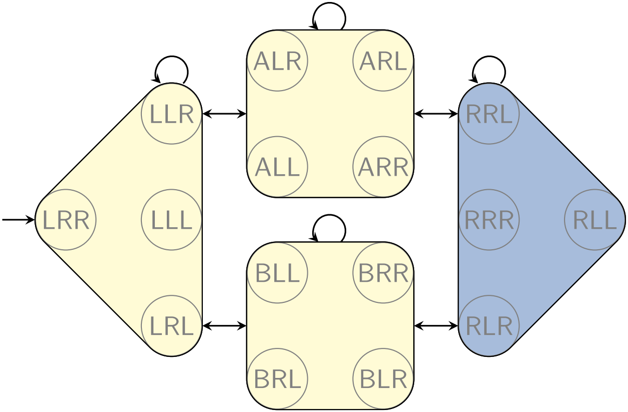


Logistics problem with one package, two trucks, two locations:

- state variable package: $\{L, R, A, B\}$
- state variable truck A: $\{L, R\}$
- state variable truck B: $\{L, R\}$

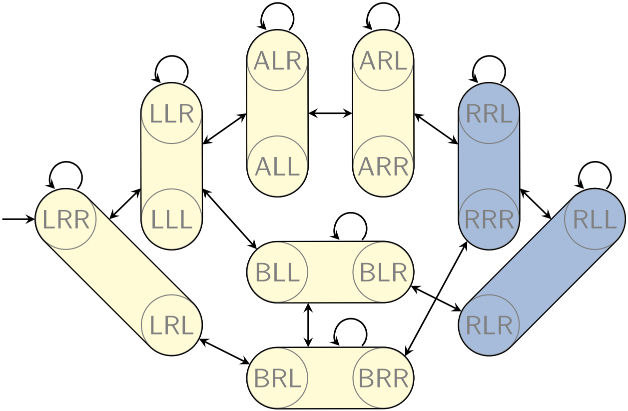
Example: Projection (1)

$\mathcal{T}^{\pi_{\{\text{package}\}}}$:



Example: Projection (2)

$\mathcal{T}^{\pi_{\{\text{package},\text{truck A}\}}}$:



Limitations of Projections

How accurate is the PDB heuristic?

- consider generalization of the example:
  N trucks, 1 package
- consider any pattern that is a proper subset of variable set V
- $h(s_0) \leq 2 \rightsquigarrow$ no better than atomic projection to package

These values cannot be improved by maximizing
over several patterns or using additive patterns.

Merge-and-shrink abstractions can represent heuristics
with $h(s_0) \geq 3$ for tasks of this kind of any size.
Time and space requirements are linear in N.

(In fact, with time/space $O(N^2)$ we can construct a merge-and-shrink abstraction that gives the perfect heuristic h^* for such tasks, but we do not show this here.)

E9.2 Main Idea

Merge-and-Shrink Abstractions: Main Idea

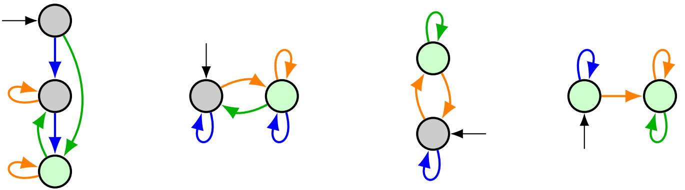
Main Idea of Merge-and-shrink Abstractions
(due to Dräger, Finkbeiner & Podelski, 2006):

Instead of perfectly reflecting a few state variables,
reflect all state variables, but in a potentially lossy way.

- Represent planning task as factored transition system (FTS): a set of (small) abstract transition systems (factors) that jointly represent the full transition system of the task.
- Iteratively transform FTS by:
  - merging: combining two factors into one
  - shrinking: reducing the size of a single factor by abstraction
- When only a single factor is left, its goal distances are the merge-and-shrink heuristic values.

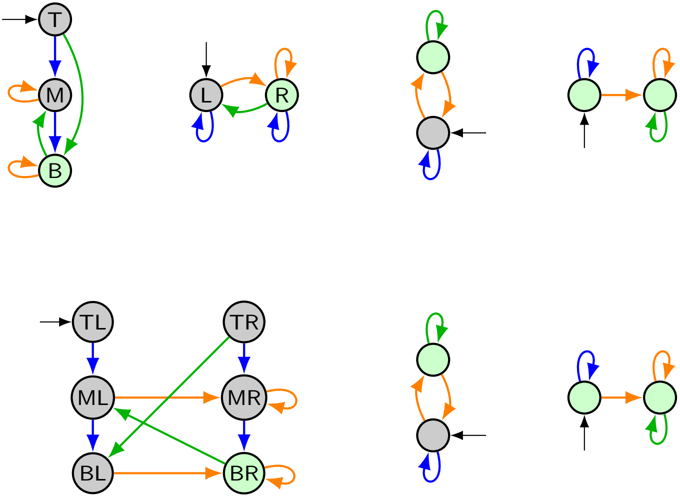
Merge-and-Shrink Abstractions: Idea

Start from atomic factors (projections to single state variables)



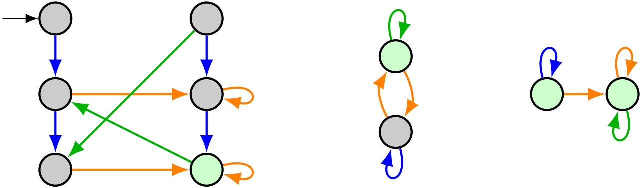
Merge-and-Shrink Abstractions: Idea

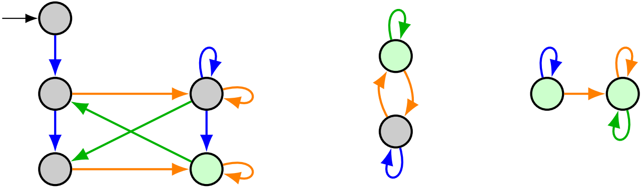
Merge: replace two factors with their product



Merge-and-Shrink Abstractions: Idea

Shrink: replace a factor by an abstraction of it





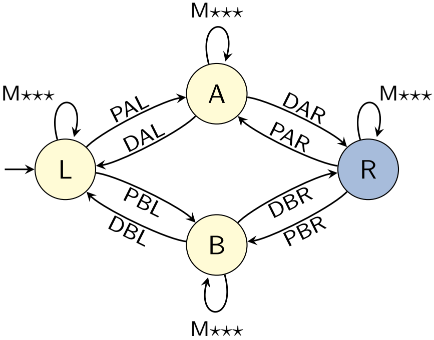
E9.3 Atomic Projections

Running Example: Explanations

- Atomic projections (projections to a single state variable) play an important role for merge-and-shrink abstractions.
- Unlike previous chapters, transition labels are critically important for merge-and-shrink.
- Hence we now look at the transition systems for atomic projections of our example task, including transition labels.
- We abbreviate labels (operator names) as in these examples:
  - MALR: move truck A from left to right
  - DAR: drop package from truck A at right location
  - PBL: pick up package with truck B at left location
- We abbreviate parallel arcs with commas and wildcards ($\star$) as in these examples:
  - PAL, DAL: two parallel arcs labeled PAL and DAL
  - MA$\star\star$: two parallel arcs labeled MALR and MARL

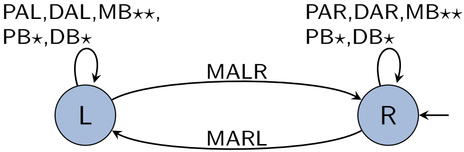
Running Example: Atomic Projection for Package

$\mathcal{T}^{\pi_{\{\text{package}\}}}$:



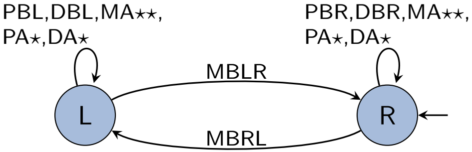
Running Example: Atomic Projection for Truck A

$\mathcal{T}^{\pi_{\{\text{truck A}\}}}$:



Running Example: Atomic Projection for Truck B

$\mathcal{T}^{\pi_{\{\text{truck B}\}}}$:



E9.4 Synchronized Product

Synchronized Product: Idea

- Given two abstract transition systems with the same labels, we can compute a product transition system.
- The product transition system captures all information of both transition systems.
- A sequence of labels is a solution for the product iff it is a solution for both factors.

Synchronized Product of Transition Systems

Definition (Synchronized Product of Transition Systems)

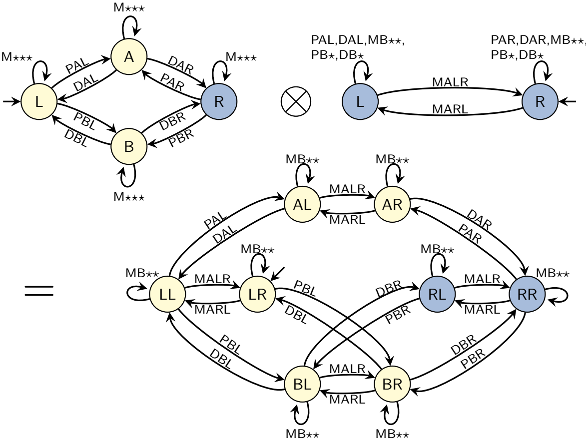
For $i \in \{1, 2\}$, let $\mathcal{T}_i = \langle S_i, L, c, T_i, s_{0i}, S_{\star i} \rangle$ be transition systems with the same labels and cost function.

The synchronized product of $\mathcal{T}_1$ and $\mathcal{T}_2$, in symbols $\mathcal{T}_1 \otimes \mathcal{T}_2$, is the transition system $\mathcal{T}_\otimes = \langle S_\otimes, L, c, T_\otimes, s_{0\otimes}, S_{\star\otimes} \rangle$ with

- $S_\otimes = S_1 \times S_2$
- $T_\otimes = \{\langle s_1, s_2 \rangle \xrightarrow{\ell} \langle t_1, t_2 \rangle \mid s_1 \xrightarrow{\ell} t_1 \in T_1 \text{ and } s_2 \xrightarrow{\ell} t_2 \in T_2\}$
- $s_{0\otimes} = \langle s_{01}, s_{02} \rangle$
- $S_{\star\otimes} = S_{\star 1} \times S_{\star 2}$

Example: Synchronized Product

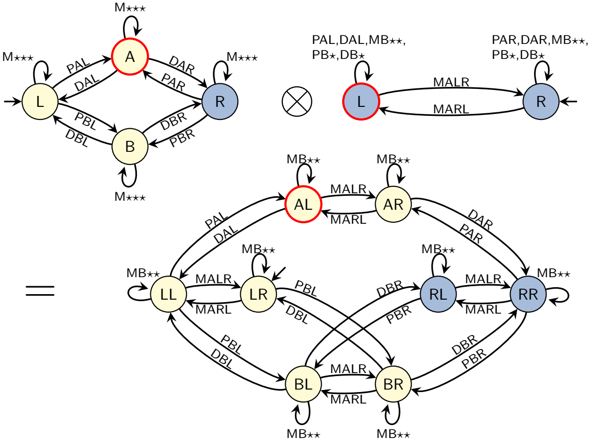
$\mathcal{T}^{\pi_{\{\text{package}\}}} \otimes \mathcal{T}^{\pi_{\{\text{truck A}\}}}$:



Example: Synchronized Product

$\mathcal{T}^{\pi_{\{\text{package}\}}} \otimes \mathcal{T}^{\pi_{\{\text{truck A}\}}}$:

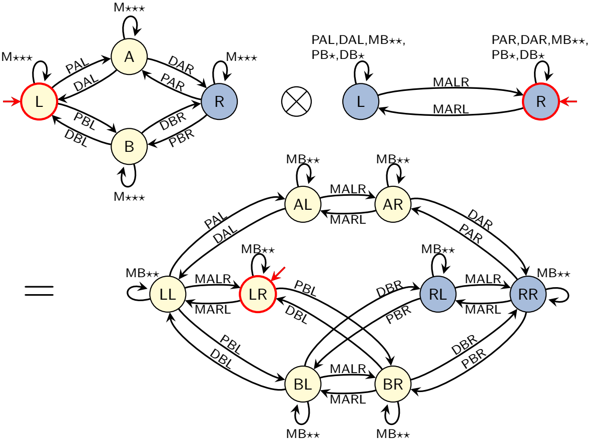
$S_\otimes = S_1 \times S_2$



Example: Synchronized Product

$\mathcal{T}^{\pi_{\{\text{package}\}}} \otimes \mathcal{T}^{\pi_{\{\text{truck A}\}}}$:

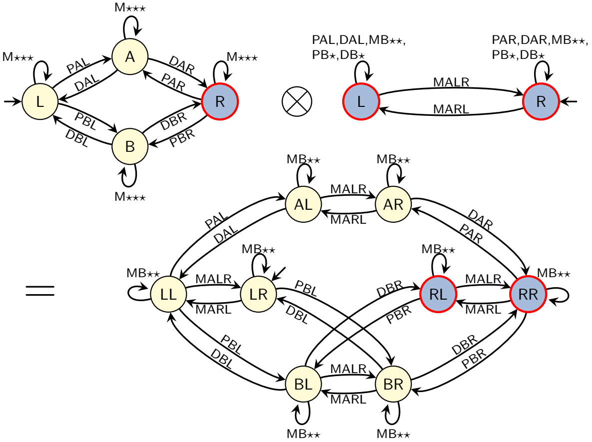
$s_{0\otimes} = \langle s_{01}, s_{02} \rangle$



Example: Synchronized Product

$\mathcal{T}^{\pi_{\{\text{package}\}}} \otimes \mathcal{T}^{\pi_{\{\text{truck A}\}}}$:

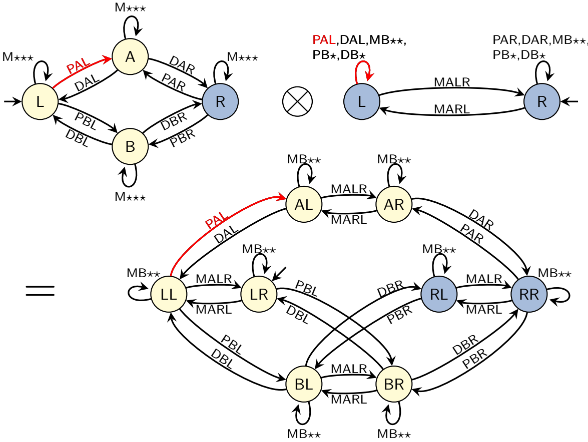
$S_{\star\otimes} = S_{\star 1} \times S_{\star 2}$



Example: Synchronized Product

$\mathcal{T}^{\pi_{\{\text{package}\}}} \otimes \mathcal{T}^{\pi_{\{\text{truck A}\}}}$:

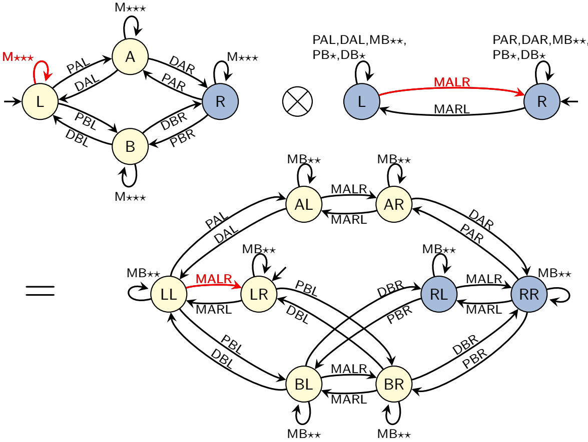
$T_{\otimes} = \{\langle s_1, s_2\rangle \xrightarrow{\ell} \langle t_1, t_2\rangle \mid s_1 \xrightarrow{\ell} t_1 \in T_1 \text{ and } s_2 \xrightarrow{\ell} t_2 \in T_2\}$



Example: Synchronized Product

$\mathcal{T}^{\pi_{\{\text{package}\}}} \otimes \mathcal{T}^{\pi_{\{\text{truck A}\}}}$:

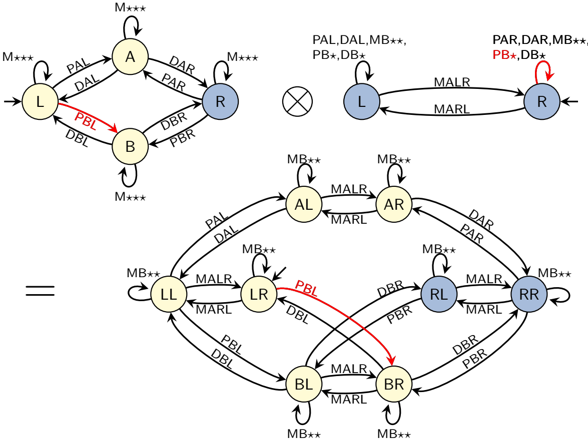
$T_\otimes = \{\langle s_1, s_2\rangle \xrightarrow{\ell} \langle t_1, t_2\rangle \mid s_1 \xrightarrow{\ell} t_1 \in T_1 \text{ and } s_2 \xrightarrow{\ell} t_2 \in T_2\}$



Example: Synchronized Product

$\mathcal{T}^{\pi_{\{\text{package}\}}} \otimes \mathcal{T}^{\pi_{\{\text{truck A}\}}}$:

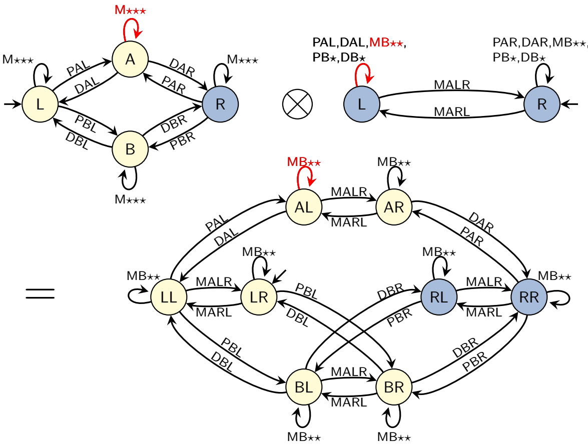
$T_\otimes = \{\langle s_1, s_2\rangle \xrightarrow{\ell} \langle t_1, t_2\rangle \mid s_1 \xrightarrow{\ell} t_1 \in T_1 \text{ and } s_2 \xrightarrow{\ell} t_2 \in T_2\}$



Example: Synchronized Product

$\mathcal{T}^{\pi_{\{\text{package}\}}} \otimes \mathcal{T}^{\pi_{\{\text{truck A}\}}}$:

$T_{\otimes} = \{\langle s_1, s_2\rangle \xrightarrow{\ell} \langle t_1, t_2\rangle \mid s_1 \xrightarrow{\ell} t_1 \in T_1 \text{ and } s_2 \xrightarrow{\ell} t_2 \in T_2\}$



Associativity and Commutativity

- Up to isomorphism ("names of states"), products are associative and commutative:
  - $(\mathcal{T} \otimes \mathcal{T}') \otimes \mathcal{T}'' \sim \mathcal{T} \otimes (\mathcal{T}' \otimes \mathcal{T}'')$
  - $\mathcal{T} \otimes \mathcal{T}' \sim \mathcal{T}' \otimes \mathcal{T}$
- We do not care about names of states and thus treat products as associative and commutative.
- We can then define the product of a set $F = \{\mathcal{T}_1, \dots, \mathcal{T}_n\}$ of transition systems: $\bigotimes F := \mathcal{T}_1 \otimes \dots \otimes \mathcal{T}_n$

E9.5 Factored Transition Systems

Factored Transition System

Definition (Factored Transition System)

A finite set $F = \{\mathcal{T}_1, \dots, \mathcal{T}_n\}$ of transition systems with the same labels and cost function is called a factored transition system (FTS).

F represents the transition system $\bigotimes F$.

A planning task gives rise to an FTS via its atomic projections:

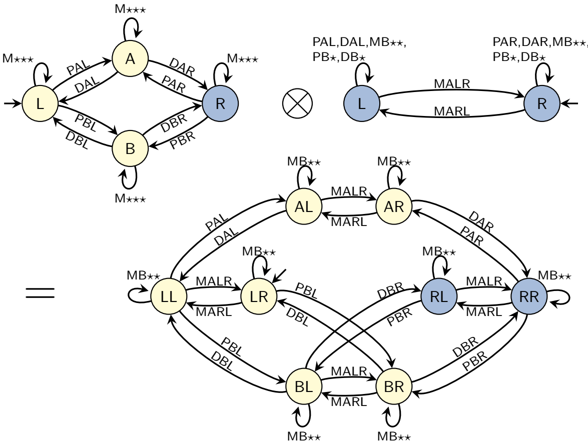
Definition (Factored Transition System Induced by Planning Task)

Let Π be a planning task with state variables V.

The factored transition system induced by Π is the FTS $F(\Pi) = \{\mathcal{T}^{\pi_{\{v\}}} \mid v \in V\}$.

Back to the Example Product

$\mathcal{T}^{\pi_{\{\text{package}\}}} \otimes \mathcal{T}^{\pi_{\{\text{truck A}\}}}$:



We have $\mathcal{T}^{\pi_{\{\text{package}\}}} \otimes \mathcal{T}^{\pi_{\{\text{truck A}\}}} \sim \mathcal{T}^{\pi_{\{\text{package,truck A}\}}}$. Coincidence?

Products of Projections

Theorem (Products of Projections)

Let Π be a SAS$^+$ planning task with variable set V, and let V_1 and V_2 be disjoint subsets of V.

Then $\mathcal{T}^{\pi_{V_1}} \otimes \mathcal{T}^{\pi_{V_2}} \sim \mathcal{T}^{\pi_{V_1 \cup V_2}}$.

(Proof omitted.)

$\rightsquigarrow$ products allow us to build finer projections from coarser ones

Recovering $\mathcal{T}(\Pi)$ from the Factored Transition System

- By repeated application of the theorem, we can recover all pattern database heuristics of a SAS$^+$ planning task as products of atomic factors.
- Moreover, by computing the product of all atomic projections, we can recover the identity abstraction $\mathrm{id} = \pi_V$.

This implies:

Corollary (Recovering $\mathcal{T}(\Pi)$ from the Factored Transition System)

Let Π be a SAS$^+$ planning task. Then $\bigotimes F(\Pi) \sim \mathcal{T}(\Pi)$.

This is an important result because it shows that $F(\Pi)$ represents all important information about Π.

E9.6 Summary

Summary

- A factored transition system is a set of transition systems that represents a larger transition system by focusing on its individual components (factors).
- For planning tasks, these factors are the atomic projections (projections to single state variables).
- The synchronized product $\mathcal{T} \otimes \mathcal{T}'$ of two transition systems with the same labels captures their "joint behaviour".
- For SAS$^+$ tasks, all projections can be obtained as products of atomic projections.
- In particular, the product of all factors of a SAS$^+$ task results in the full transition system of the task.