Planning and Optimization

E4. Abstractions: Formal Definition and Heuristics

Malte Helmert and Gabriele Röger

Universität Basel

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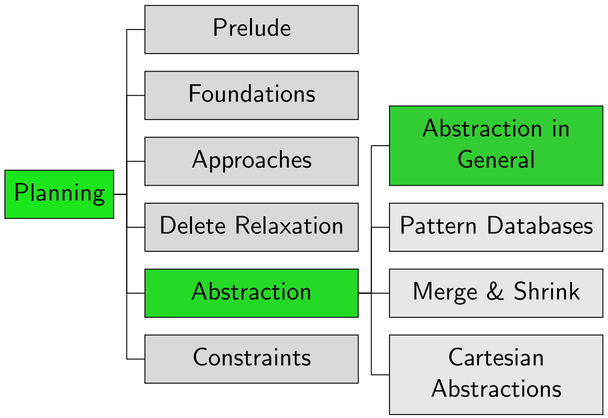
Content of the Course



E4.1 Reminder: Transition Systems

Transition Systems

Reminder from Chapter B1:

Definition (Transition System)

A transition system is a 6-tuple $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$ where

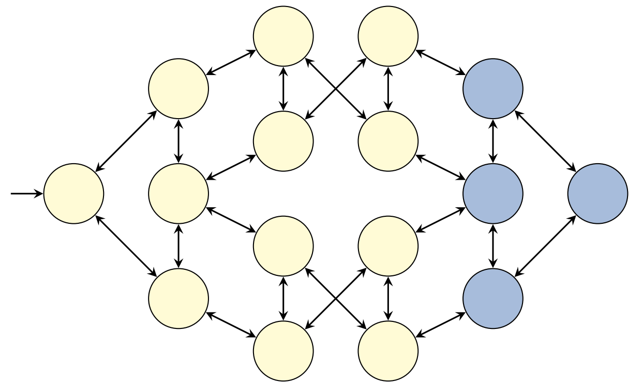
- S is a finite set of states,
- L is a finite set of (transition) labels,
- $c : L \to \mathbb{R}_0^+$ is a label cost function,
- $T \subseteq S \times L \times S$ is the transition relation,
- $s_0 \in S$ is the initial state, and
- $S_\star \subseteq S$ is the set of goal states.

We say that $\mathcal{T}$ has the transition $\langle s, \ell, s' \rangle$ if $\langle s, \ell, s' \rangle \in T$.

We also write this as $s \xrightarrow{\ell} s'$, or $s \to s'$ when not interested in ℓ.

Note: Transition systems are also called state spaces.

Transition Systems: Example



Note: To reduce clutter, our figures often omit arc labels and costs and collapse transitions between identical states. However, these are important for the formal definition of the transition system.

Mapping Planning Tasks to Transition Systems

Reminder from Chapters B3 and E1:

Definition (Transition System Induced by a Planning Task)

The planning task $\Pi = \langle V, I, O, \gamma \rangle$ induces the transition system $\mathcal{T}(\Pi) = \langle S, L, c, T, s_0, S_\star \rangle$, where

- S is the set of all states over state variables V,
- L is the set of operators O,
- $c(o) = cost(o)$ for all operators $o \in O$,
- $T = \{\langle s, o, s' \rangle \mid s \in S,\ o \text{ applicable in } s,\ s' = s[\![o]\!]\}$,
- $s_0 = I$, and
- $S_\star = \{s \in S \mid s \models \gamma\}$.

(same definition for propositional and finite-domain representation)

Tasks in Finite-Domain Representation

Notes:

- We will focus on planning tasks in finite-domain representation (FDR) while studying abstractions.
- All concepts apply equally to propositional planning tasks.
- However, FDR tasks are almost always used by algorithms in this context because they tend to have fewer useless (physically impossible) states.
- Useless states can hurt the efficiency of abstraction-based algorithms.

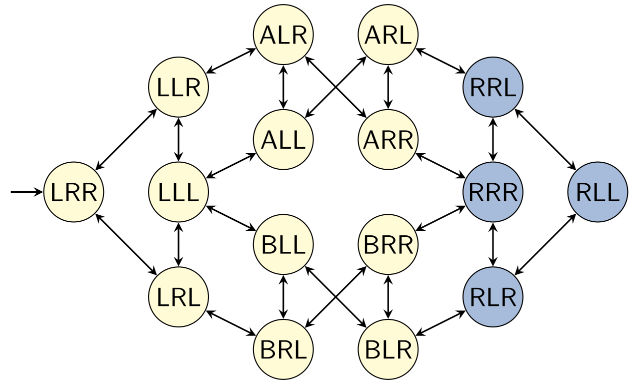
Example Task: One Package, Two Trucks

Example (One Package, Two Trucks)

Consider the following FDR planning task $\langle V, I, O, \gamma\rangle$:

- $V = \{p, t_{\mathsf{A}}, t_{\mathsf{B}}\}$ with
  - $\mathrm{dom}(p) = \{\mathsf{L}, \mathsf{R}, \mathsf{A}, \mathsf{B}\}$
  - $\mathrm{dom}(t_{\mathsf{A}}) = \mathrm{dom}(t_{\mathsf{B}}) = \{\mathsf{L}, \mathsf{R}\}$
- $I = \{p \mapsto \mathsf{L}, t_{\mathsf{A}} \mapsto \mathsf{R}, t_{\mathsf{B}} \mapsto \mathsf{R}\}$
- $O = \{\mathsf{pickup}_{i,j} \mid i \in \{\mathsf{A}, \mathsf{B}\}, j \in \{\mathsf{L}, \mathsf{R}\}\}$
  $\cup \{\mathsf{drop}_{i,j} \mid i \in \{\mathsf{A}, \mathsf{B}\}, j \in \{\mathsf{L}, \mathsf{R}\}\}$
  $\cup \{\mathsf{move}_{i,j,j'} \mid i \in \{\mathsf{A}, \mathsf{B}\}, j, j' \in \{\mathsf{L}, \mathsf{R}\}, j \neq j'\}$, where
  - $\mathsf{pickup}_{i,j} = \langle t_i = j \land p = j, p := i, 1\rangle$
  - $\mathsf{drop}_{i,j} = \langle t_i = j \land p = i, p := j, 1\rangle$
  - $\mathsf{move}_{i,j,j'} = \langle t_i = j, t_i := j', 1\rangle$
- $\gamma = (p = \mathsf{R})$

Transition System of Example Task



- State $\{p \mapsto i, t_{\mathsf{A}} \mapsto j, t_{\mathsf{B}} \mapsto k\}$ is depicted as ijk.
- Transition labels are again not shown. For example, the transition from LLL to ALL has the label $\mathsf{pickup}_{\mathsf{A},\mathsf{L}}$.

E4.2 Abstractions

Abstractions

Definition (Abstraction)

Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star\rangle$ be a transition system.

An abstraction (also: abstraction function, abstraction mapping) of $\mathcal{T}$ is a function $\alpha : S \to S^\alpha$ defined on the states of $\mathcal{T}$, where S^α is an arbitrary set.

Without loss of generality, we require that α is surjective.

Intuition: α maps the states of $\mathcal{T}$ to another (usually smaller) abstract state space.

Abstract Transition System

Definition (Abstract Transition System)

Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$ be a transition system, and let $\alpha : S \to S^\alpha$ be an abstraction of $\mathcal{T}$.

The abstract transition system induced by α, in symbols $\mathcal{T}^\alpha$, is the transition system $\mathcal{T}^\alpha = \langle S^\alpha, L, c, T^\alpha, s_0^\alpha, S_\star^\alpha \rangle$ defined by:

- $T^\alpha = \{\langle \alpha(s), \ell, \alpha(t) \rangle \mid \langle s, \ell, t \rangle \in T\}$
- $s_0^\alpha = \alpha(s_0)$
- $S_\star^\alpha = \{\alpha(s) \mid s \in S_\star\}$

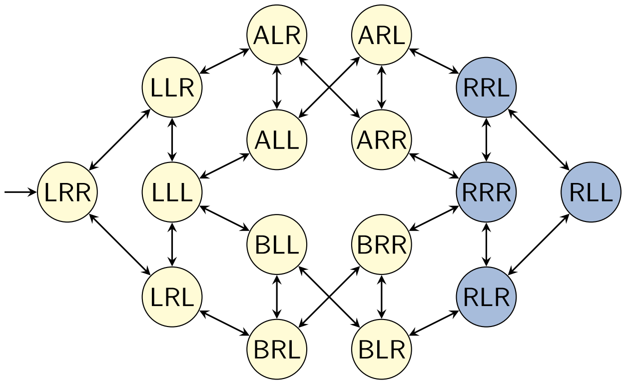
Concrete and Abstract State Space

Let $\mathcal{T}$ be a transition system and α be an abstraction of $\mathcal{T}$.

- $\mathcal{T}$ is called the concrete transition system.
- $\mathcal{T}^\alpha$ is called the abstract transition system.
- Similarly: concrete/abstract state space, concrete/abstract transition, etc.

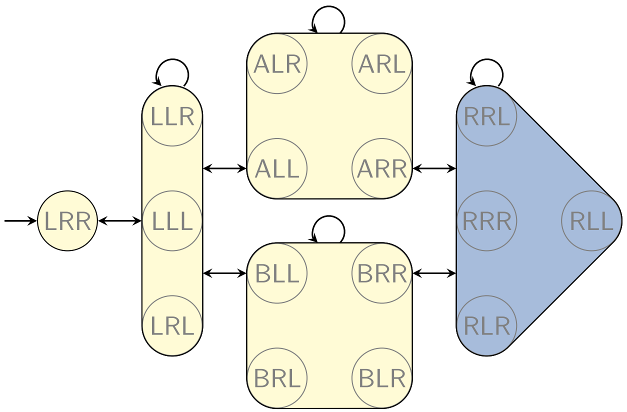
Abstraction: Example

concrete transition system



Abstraction: Example

abstract transition system



Note: Most arcs represent many parallel transitions.

Strict Homomorphisms

- The abstraction mapping α that transforms $\mathcal{T}$ to $\mathcal{T}^\alpha$ is also called a strict homomorphism from $\mathcal{T}$ to $\mathcal{T}^\alpha$.
- Roughly speaking, in mathematics a homomorphism is a property-preserving mapping between structures.
- A strict homomorphism is one where no additional features are introduced. A non-strict homomorphism in planning would mean that the abstract transition system may include additional transitions and goal states not induced by α.

E4.3 Abstraction Heuristics

Abstraction Heuristics

Definition (Abstraction Heuristic)

Let $\alpha : S \to S^\alpha$ be an abstraction of a transition system $\mathcal{T}$.

The abstraction heuristic induced by α, written h^α, is the heuristic function $h^\alpha : S \to \mathbb{R}_0^+ \cup \{\infty\}$ defined as

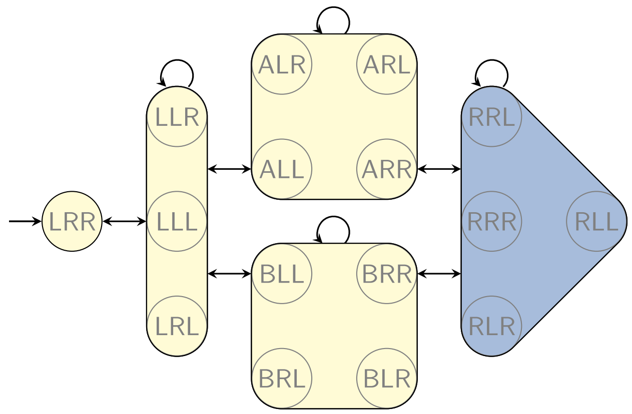
$$h^\alpha(s) = h^*_{\mathcal{T}^\alpha}(\alpha(s)) \quad \text{for all } s \in S,$$

where $h^*_{\mathcal{T}^\alpha}$ denotes the goal distance function in $\mathcal{T}^\alpha$.

Notes:

- $h^\alpha(s) = \infty$ if no goal state of $\mathcal{T}^\alpha$ is reachable from $\alpha(s)$
- We also apply abstraction terminology to planning tasks Π, which stand for their induced transition systems. For example, an abstraction of Π is an abstraction of $\mathcal{T}(\Pi)$.

Abstraction Heuristics: Example



$h^\alpha(\{p \mapsto \mathsf{L}, t_\mathsf{A} \mapsto \mathsf{R}, t_\mathsf{B} \mapsto \mathsf{R}\}) = 3$

Consistency of Abstraction Heuristics (1)

Theorem (Consistency and Admissibility of h^α)

Let α be an abstraction of a transition system $\mathcal{T}$.
Then h^α is safe, goal-aware, admissible and consistent.

Proof.

We prove goal-awareness and consistency;
the other properties follow from these two.

Let $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$.
Let $\mathcal{T}^\alpha = \langle S^\alpha, L, c, T^\alpha, s_0^\alpha, S_\star^\alpha \rangle$.

Goal-awareness: We need to show that $h^\alpha(s) = 0$ for all $s \in S_\star$, so let $s \in S_\star$. Then $\alpha(s) \in S_\star^\alpha$ by the definition of abstract transition systems, and hence $h^\alpha(s) = h^*_{\mathcal{T}^\alpha}(\alpha(s)) = 0$. ...

Consistency of Abstraction Heuristics (2)

Proof (continued).

Consistency: Consider any state transition $s \xrightarrow{\ell} t$ of $\mathcal{T}$.
We need to show $h^\alpha(s) \leq c(\ell) + h^\alpha(t)$.

By the definition of $\mathcal{T}^\alpha$, we get $\alpha(s) \xrightarrow{\ell} \alpha(t) \in T^\alpha$.
Hence, $\alpha(t)$ is a successor of $\alpha(s)$ in $\mathcal{T}^\alpha$ via the label ℓ.

We get:

$$\begin{aligned} h^\alpha(s) &= h^*_{\mathcal{T}^\alpha}(\alpha(s)) \\ &\leq c(\ell) + h^*_{\mathcal{T}^\alpha}(\alpha(t)) \\ &= c(\ell) + h^\alpha(t), \end{aligned}$$

where the inequality holds because perfect goal distances $h^*_{\mathcal{T}^\alpha}$ are consistent in $\mathcal{T}^\alpha$.
(The shortest path from $\alpha(s)$ to the goal in $\mathcal{T}^\alpha$ cannot be longer than the shortest path from $\alpha(s)$ to the goal via $\alpha(t)$.) $\square$

E4.4 Coarsenings and Refinements

Abstractions of Abstractions

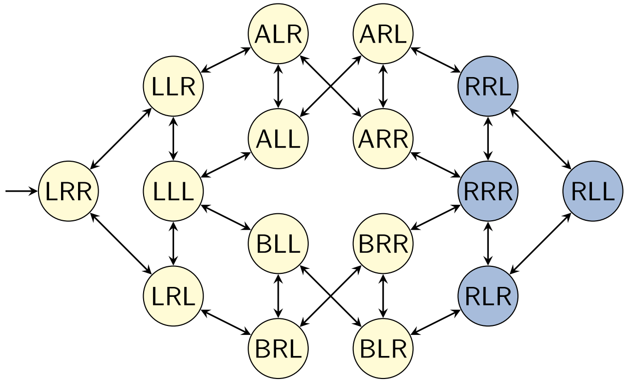
Since abstractions map transition systems to transition systems, they are composable:

- Using a first abstraction $\alpha : S \to S'$, map $\mathcal{T}$ to $\mathcal{T}^\alpha$.
- Using a second abstraction $\beta : S' \to S''$, map $\mathcal{T}^\alpha$ to $(\mathcal{T}^\alpha)^\beta$.

The result is the same as directly using the abstraction $(\beta \circ \alpha)$:

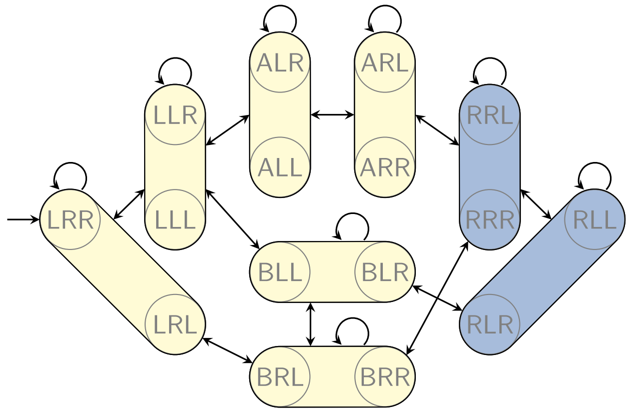
- Let $\gamma : S \to S''$ be defined as $\gamma(s) = (\beta \circ \alpha)(s) = \beta(\alpha(s))$.
- Then $\mathcal{T}^\gamma = (\mathcal{T}^\alpha)^\beta$.

Abstractions of Abstractions: Example (1)



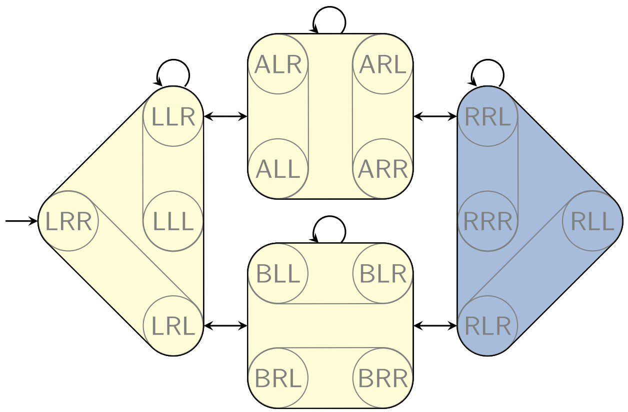
transition system $\mathcal{T}$

Abstractions of Abstractions: Example (2)



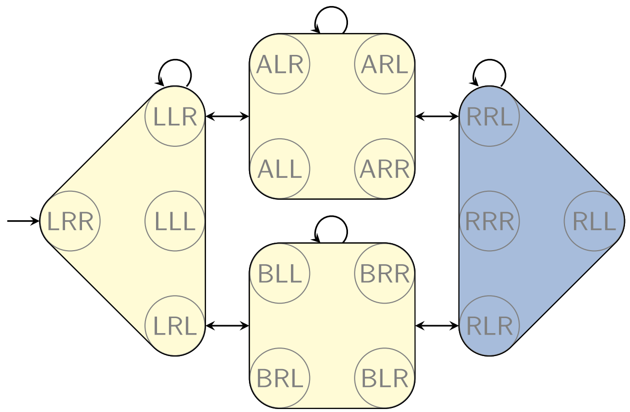
Transition system $\mathcal{T}'$ as an abstraction of $\mathcal{T}$

Abstractions of Abstractions: Example (3)



Transition system $\mathcal{T}''$ as an abstraction of $\mathcal{T}'$

Abstractions of Abstractions: Example (3)



Transition system $\mathcal{T}''$ as an abstraction of $\mathcal{T}$

Coarsenings and Refinements

Definition (Coarsening and Refinement)

Let α and γ be abstractions of the same transition system such that $\gamma = \beta \circ \alpha$ for some function β.

Then γ is called a coarsening of α and α is called a refinement of γ.

Heuristic Quality of Refinements

Theorem (Heuristic Quality of Refinements)

Let α and γ be abstractions of the same transition system such that α is a refinement of γ.

Then h^α dominates h^γ.

In other words, $h^\gamma(s) \leq h^\alpha(s) \leq h^*(s)$ for all states s.

Heuristic Quality of Refinements: Proof

Proof.

Since α is a refinement of γ, there exists a function β with $\gamma = \beta \circ \alpha$.

For all states s of Π, we get:

$$\begin{aligned} h^\gamma(s) &= h^*_{\mathcal{T}^\gamma}(\gamma(s)) \\ &= h^*_{\mathcal{T}^\gamma}(\beta(\alpha(s))) \\ &= h^\beta_{\mathcal{T}^\alpha}(\alpha(s)) \\ &\leq h^*_{\mathcal{T}^\alpha}(\alpha(s)) \\ &= h^\alpha(s), \end{aligned}$$

where the inequality holds because $h^\beta_{\mathcal{T}^\alpha}$ is an admissible heuristic in the transition system $\mathcal{T}^\alpha$. □

E4.5 Summary

Summary

- An abstraction is a function α that maps the states S of a transition system to another (usually smaller) set S^α.
- This induces an abstract transition system $\mathcal{T}^\alpha$, which behaves like the original transition system $\mathcal{T}$ except that states mapped to the same abstract state cannot be distinguished.
- Abstractions α induce abstraction heuristics h^α: $h^\alpha(s)$ is the goal distance of $\alpha(s)$ in the abstract transition system.
- Abstraction heuristics are safe, goal-aware, admissible and consistent.
- Abstractions can be composed, leading to coarser vs. finer abstractions. Heuristics for finer abstractions dominate those for coarser ones.