

# Planning and Optimization

## E3. Abstractions: Introduction

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## E3.1 Introduction

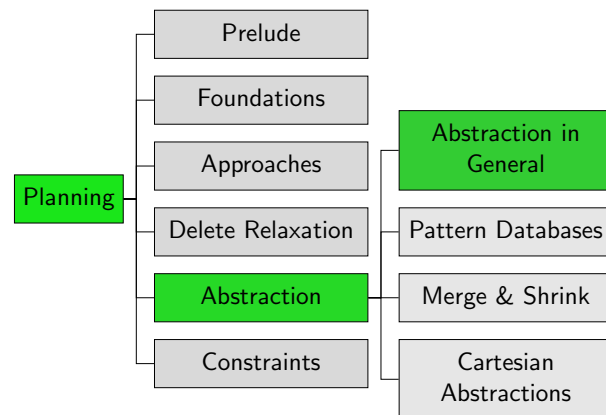
## E3.2 Practical Requirements

## E3.3 Multiple Abstractions

## E3.4 Outlook

## E3.5 Summary

## Content of the Course



## E3.1 Introduction

## Coming Up with Heuristics in a Principled Way

### General Procedure for Obtaining a Heuristic

Solve a simplified version of the problem.

Major ideas for heuristics in the planning literature:

- ▶ delete relaxation
- ▶ **abstraction**
- ▶ critical paths
- ▶ landmarks
- ▶ network flows
- ▶ potential heuristics

Heuristics based on **abstraction** are among the most prominent techniques for **optimal planning**.

## Abstracting a Transition System

Abstracting a transition system means **dropping some distinctions** between states, while **preserving the transition behaviour** as much as possible.

- ▶ An abstraction of a transition system  $\mathcal{T}$  is defined by an **abstraction mapping**  $\alpha$  that defines which states of  $\mathcal{T}$  should be distinguished and which ones should not.
- ▶ From  $\mathcal{T}$  and  $\alpha$ , we compute an **abstract transition system**  $\mathcal{T}^\alpha$  which is similar to  $\mathcal{T}$ , but smaller.
- ▶ The **abstract goal distances** (goal distances in  $\mathcal{T}^\alpha$ ) are used as heuristic estimates for goal distances in  $\mathcal{T}$ .

## Abstracting a Transition System: Example

example from domain-specific heuristic search:

### Example (15-Puzzle)

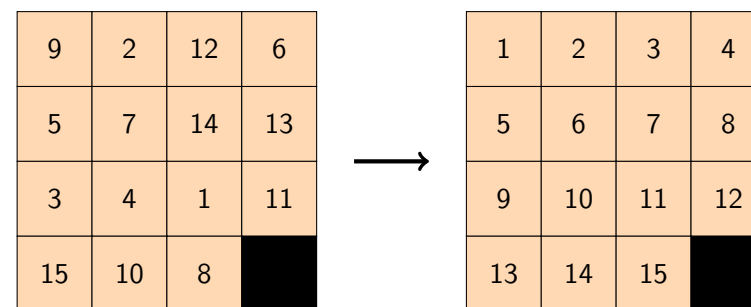
A **15-puzzle** state is given by a permutation  $\langle b, t_1, \dots, t_{15} \rangle$  of  $\{1, \dots, 16\}$ , where  $b$  denotes the blank position and the other components denote the positions of the 15 tiles.

One possible **abstraction mapping** ignores the precise location of tiles 8–15, i.e., two states are distinguished iff they differ in the position of the blank or one of the tiles 1–7:

$$\alpha(\langle b, t_1, \dots, t_{15} \rangle) = \langle b, t_1, \dots, t_7 \rangle$$

The heuristic values for this abstraction correspond to the cost of moving tiles 1–7 to their goal positions.

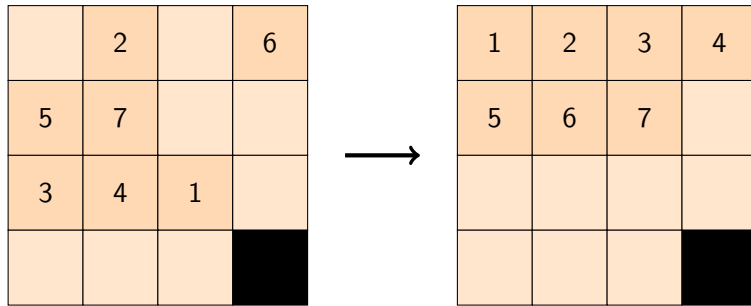
## Abstraction Example: 15-Puzzle



**real state space:**

- ▶  $16! = 20922789888000 \approx 2 \cdot 10^{13}$  states
- ▶  $\frac{16!}{2} = 10461394944000 \approx 10^{13}$  reachable states

## Abstraction Example: 15-Puzzle



abstract state space:

- ▶  $16 \cdot 15 \cdot \dots \cdot 9 = 518918400 \approx 5 \cdot 10^8$  states
- ▶  $16 \cdot 15 \cdot \dots \cdot 9 = 518918400 \approx 5 \cdot 10^8$  reachable states

## Computing the Abstract Transition System

Given  $\mathcal{T}$  and  $\alpha$ , how do we compute  $\mathcal{T}^\alpha$ ?

### Requirement

We want to obtain an **admissible heuristic**.

Hence,  $h^*(\alpha(s))$  (in the abstract state space  $\mathcal{T}^\alpha$ ) should never overestimate  $h^*(s)$  (in the concrete state space  $\mathcal{T}$ ).

An easy way to achieve this is to ensure that **all solutions in  $\mathcal{T}$  are also present in  $\mathcal{T}^\alpha$** :

- ▶ If  $s$  is a goal state in  $\mathcal{T}$ , then  $\alpha(s)$  is a goal state in  $\mathcal{T}^\alpha$ .
- ▶ If  $\mathcal{T}$  has a transition from  $s$  to  $t$ , then  $\mathcal{T}^\alpha$  has a transition from  $\alpha(s)$  to  $\alpha(t)$  with the same cost.

## Computing the Abstract Transition System: Example

### Example (15-Puzzle)

In the running example:

- ▶  $\mathcal{T}$  has the unique goal state  $\langle 16, 1, 2, \dots, 15 \rangle$ .  
 $\leadsto \mathcal{T}^\alpha$  has the unique goal state  $\langle 16, 1, 2, \dots, 7 \rangle$ .
- ▶ Let  $x$  and  $y$  be neighbouring positions in the  $4 \times 4$  grid.  
 $\mathcal{T}$  has a transition from  $\langle x, t_1, \dots, t_{i-1}, y, t_{i+1}, \dots, t_{15} \rangle$   
to  $\langle y, t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_{15} \rangle$  for all  $i \in \{1, \dots, 15\}$ .  
 $\leadsto \mathcal{T}^\alpha$  has a transition from  $\langle x, t_1, \dots, t_{i-1}, y, t_{i+1}, \dots, t_7 \rangle$   
to  $\langle y, t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_7 \rangle$  for all  $i \in \{1, \dots, 7\}$ .  
 $\leadsto$  Moreover,  $\mathcal{T}^\alpha$  has a transition from  $\langle x, t_1, \dots, t_7 \rangle$   
to  $\langle y, t_1, \dots, t_7 \rangle$  if  $y \notin \{t_1, \dots, t_7\}$ .

## E3.2 Practical Requirements

## Practical Requirements for Abstractions

To be useful in practice, an abstraction heuristic must be efficiently computable. This gives us two requirements for  $\alpha$ :

- ▶ For a given state  $s$ , the **abstract state**  $\alpha(s)$  must be efficiently computable.
- ▶ For a given abstract state  $\alpha(s)$ , the **abstract goal distance**  $h^*(\alpha(s))$  must be efficiently computable.

There are a number of ways of achieving these requirements:

- ▶ **pattern database heuristics** (Culberson & Schaeffer, 1996)
- ▶ domain abstractions (Hernádvölgyi and Holte, 2000)
- ▶ **merge-and-shrink abstractions** (Dräger, Finkbeiner & Podelski, 2006)
- ▶ **Cartesian abstractions** (Ball, Podelski & Rajamani, 2001)
- ▶ structural patterns (Katz & Domshlak, 2008)

## Practical Requirements for Abstractions: Example

### Example (15-Puzzle)

In our running example,  $\alpha$  can be very efficiently computed: just project the given 16-tuple to its first 8 components.

To compute abstract goal distances efficiently during search, the most common approach is to precompute **all abstract goal distances** prior to search by performing a backward uniform-cost search from the abstract goal state(s). These distances are then stored in a table (requires  $\approx 495$  MiB RAM).

During search, computing  $h^*(\alpha(s))$  is just a table lookup.

This heuristic is an example of a **pattern database heuristic**.

## E3.3 Multiple Abstractions

## Multiple Abstractions

- ▶ One important practical question is how to come up with a suitable abstraction mapping  $\alpha$ .
- ▶ Indeed, there is usually a **huge number of possibilities**, and it is important to pick good abstractions (i.e., ones that lead to informative heuristics).
- ▶ However, it is generally **not necessary to commit to a single abstraction**.

## Combining Multiple Abstractions

### Maximizing several abstractions:

- ▶ Each abstraction mapping gives rise to an admissible heuristic.
- ▶ By computing the **maximum** of several admissible heuristics, we obtain another admissible heuristic which **dominates** the component heuristics.
- ▶ Thus, we can always compute several abstractions and maximize over the individual abstract goal distances.

### Adding several abstractions:

- ▶ In some cases, we can even compute the **sum** of individual estimates and still stay admissible.
- ▶ Summation often leads to **much higher estimates** than maximization, so it is important to understand **under which conditions** summation of heuristics is **admissible**.

## Maximizing Several Abstractions: Example

### Example (15-Puzzle)

- ▶ mapping to tiles 1–7 was arbitrary  
~> can use **any subset** of tiles
- ▶ with the same amount of memory required for the tables for the mapping to tiles 1–7, we could store the tables for **nine different abstractions** to six tiles and the blank
- ▶ use **maximum** of individual estimates

## Adding Several Abstractions: Example

|    |    |    |    |
|----|----|----|----|
| 9  | 2  | 12 | 6  |
| 5  | 7  | 14 | 13 |
| 3  | 4  | 1  | 11 |
| 15 | 10 | 8  |    |

|    |    |    |    |
|----|----|----|----|
| 9  | 2  | 12 | 6  |
| 5  | 7  | 14 | 13 |
| 3  | 4  | 1  | 11 |
| 15 | 10 | 8  |    |

- ▶ **1st abstraction**: ignore precise location of 8–15
- ▶ **2nd abstraction**: ignore precise location of 1–7
- ~> Is the **sum** of the abstraction heuristics **admissible**?

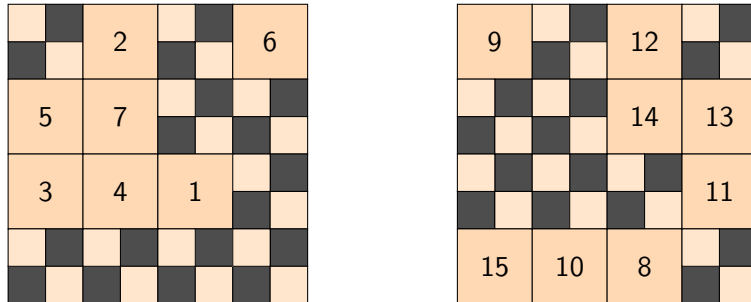
## Adding Several Abstractions: Example

|   |   |   |   |
|---|---|---|---|
|   | 2 |   | 6 |
| 5 | 7 |   |   |
| 3 | 4 | 1 |   |
|   |   |   |   |

|    |    |    |    |
|----|----|----|----|
| 9  |    | 12 |    |
|    |    | 14 | 13 |
|    |    |    | 11 |
| 15 | 10 | 8  |    |

- ▶ **1st abstraction**: ignore precise location of 8–15
- ▶ **2nd abstraction**: ignore precise location of 1–7
- ~> The **sum** of the abstraction heuristics is **not admissible**.

## Adding Several Abstractions: Example



- ▶ **1st abstraction:** ignore precise location of 8–15 and blank
- ▶ **2nd abstraction:** ignore precise location of 1–7 and blank
- ~> The **sum** of the abstraction heuristics is **admissible**.

## E3.4 Outlook

## Our Plan for the Next Lectures

In the following, we take a deeper look at abstractions and their use for admissible heuristics.

In the next two chapters, we **formally introduce** abstractions and abstraction heuristics and study some of their most important properties.

Afterwards, we discuss some particular classes of abstraction heuristics in detail, namely

- ▶ **pattern database heuristics,**
- ▶ **merge-and-shrink abstractions** and
- ▶ **Cartesian abstractions.**

## E3.5 Summary

## Summary

- ▶ **Abstraction** is one of the principled ways of deriving heuristics for planning tasks and transition systems in general.
- ▶ The key idea is to map states to a smaller **abstract transition system**  $\mathcal{T}^\alpha$  by means of an **abstraction function**  $\alpha$ .
- ▶ **Goal distances in  $\mathcal{T}^\alpha$**  are then used as **admissible estimates** for goal distances in the original transition system.
- ▶ To be **practical**, we must be able to **compute abstraction functions** and **determine abstract goal distances efficiently**.
- ▶ Often, **multiple abstractions** are used.  
They can always be **maximized** admissibly.
- ▶ **Adding** abstraction heuristics is not always admissible.  
When it is, it leads to a stronger heuristic than maximizing.