

Planning and Optimization

E3. Abstractions: Introduction

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November 5, 2025

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E3.1 Introduction

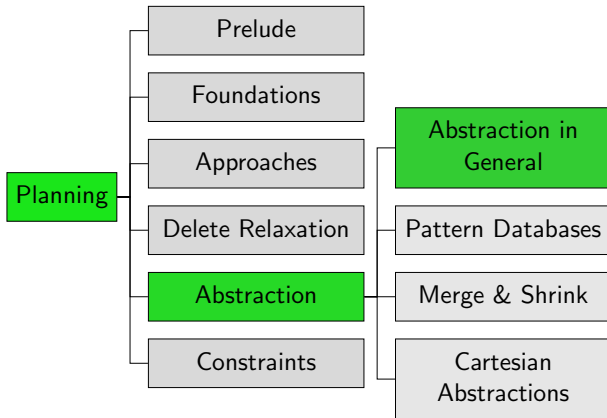
E3.2 Practical Requirements

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Content of the Course



E3.1 Introduction

Coming Up with Heuristics in a Principled Way

General Procedure for Obtaining a Heuristic

Solve a simplified version of the problem.

Major ideas for heuristics in the planning literature:

- ▶ delete relaxation
- ▶ **abstraction**
- ▶ critical paths
- ▶ landmarks
- ▶ network flows
- ▶ potential heuristics

Heuristics based on **abstraction** are among the most prominent techniques for **optimal planning**.

Abstracting a Transition System

Abstracting a transition system means **dropping some distinctions** between states, while **preserving the transition behaviour** as much as possible.

- ▶ An abstraction of a transition system $\mathcal{T}$ is defined by an **abstraction mapping** α that defines which states of $\mathcal{T}$ should be distinguished and which ones should not.
- ▶ From $\mathcal{T}$ and α , we compute an **abstract transition system** $\mathcal{T}^\alpha$ which is similar to $\mathcal{T}$, but smaller.
- ▶ The **abstract goal distances** (goal distances in $\mathcal{T}^\alpha$) are used as heuristic estimates for goal distances in $\mathcal{T}$.

Abstracting a Transition System: Example

example from domain-specific heuristic search:

Example (15-Puzzle)

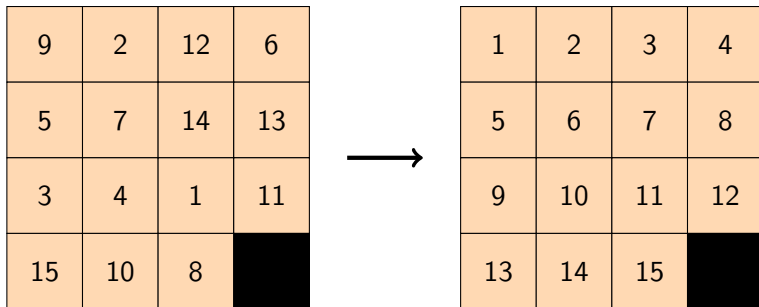
A **15-puzzle** state is given by a permutation $\langle b, t_1, \dots, t_{15} \rangle$ of $\{1, \dots, 16\}$, where b denotes the blank position and the other components denote the positions of the 15 tiles.

One possible **abstraction mapping** ignores the precise location of tiles 8–15, i.e., two states are distinguished iff they differ in the position of the blank or one of the tiles 1–7:

$$\alpha(\langle b, t_1, \dots, t_{15} \rangle) = \langle b, t_1, \dots, t_7 \rangle$$

The heuristic values for this abstraction correspond to the cost of moving tiles 1–7 to their goal positions.

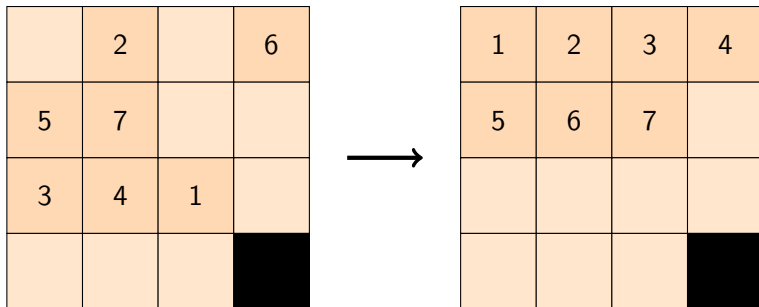
Abstraction Example: 15-Puzzle



real state space:

- ▶ $16! = 20922789888000 \approx 2 \cdot 10^{13}$ states
- ▶ $\frac{16!}{2} = 10461394944000 \approx 10^{13}$ reachable states

Abstraction Example: 15-Puzzle



abstract state space:

- ▶ $16 \cdot 15 \cdot \dots \cdot 9 = 518918400 \approx 5 \cdot 10^8$ states
- ▶ $16 \cdot 15 \cdot \dots \cdot 9 = 518918400 \approx 5 \cdot 10^8$ reachable states

Computing the Abstract Transition System

Given $\mathcal{T}$ and α , how do we compute $\mathcal{T}^\alpha$?

Requirement

We want to obtain an **admissible heuristic**.

Hence, $h^*(\alpha(s))$ (in the abstract state space $\mathcal{T}^\alpha$) should never overestimate $h^*(s)$ (in the concrete state space $\mathcal{T}$).

An easy way to achieve this is to ensure that **all solutions in $\mathcal{T}$ are also present in $\mathcal{T}^\alpha$** :

- ▶ If s is a goal state in $\mathcal{T}$, then $\alpha(s)$ is a goal state in $\mathcal{T}^\alpha$.
- ▶ If $\mathcal{T}$ has a transition from s to t , then $\mathcal{T}^\alpha$ has a transition from $\alpha(s)$ to $\alpha(t)$ with the same cost.

Computing the Abstract Transition System: Example

Example (15-Puzzle)

In the running example:

- ▶ $\mathcal{T}$ has the unique goal state $\langle 16, 1, 2, \dots, 15 \rangle$.
 - ↪ $\mathcal{T}^\alpha$ has the unique goal state $\langle 16, 1, 2, \dots, 7 \rangle$.
- ▶ Let x and y be neighbouring positions in the 4×4 grid.
 $\mathcal{T}$ has a transition from $\langle x, t_1, \dots, t_{i-1}, y, t_{i+1}, \dots, t_{15} \rangle$
to $\langle y, t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_{15} \rangle$ for all $i \in \{1, \dots, 15\}$.
 - ↪ $\mathcal{T}^\alpha$ has a transition from $\langle x, t_1, \dots, t_{i-1}, y, t_{i+1}, \dots, t_7 \rangle$
to $\langle y, t_1, \dots, t_{i-1}, x, t_{i+1}, \dots, t_7 \rangle$ for all $i \in \{1, \dots, 7\}$.
 - ↪ Moreover, $\mathcal{T}^\alpha$ has a transition from $\langle x, t_1, \dots, t_7 \rangle$
to $\langle y, t_1, \dots, t_7 \rangle$ if $y \notin \{t_1, \dots, t_7\}$.

E3.2 Practical Requirements

Practical Requirements for Abstractions

To be useful in practice, an abstraction heuristic must be efficiently computable. This gives us two requirements for α :

- ▶ For a given state s , the **abstract state** $\alpha(s)$ must be efficiently computable.
- ▶ For a given abstract state $\alpha(s)$, the **abstract goal distance** $h^*(\alpha(s))$ must be efficiently computable.

There are a number of ways of achieving these requirements:

- ▶ **pattern database heuristics** (Culberson & Schaeffer, 1996)
- ▶ domain abstractions (Hernádvölgyi and Holte, 2000)
- ▶ **merge-and-shrink abstractions** (Dräger, Finkbeiner & Podelski, 2006)
- ▶ **Cartesian abstractions** (Ball, Podelski & Rajamani, 2001)
- ▶ structural patterns (Katz & Domshlak, 2008)

Practical Requirements for Abstractions: Example

Example (15-Puzzle)

In our running example, α can be very efficiently computed: just project the given 16-tuple to its first 8 components.

To compute abstract goal distances efficiently during search, the most common approach is to precompute **all abstract goal distances** prior to search by performing a backward uniform-cost search from the abstract goal state(s). These distances are then stored in a table (requires ≈ 495 MiB RAM).

During search, computing $h^*(\alpha(s))$ is just a table lookup.

This heuristic is an example of a **pattern database heuristic**.

E3.3 Multiple Abstractions

Multiple Abstractions

- ▶ One important practical question is how to come up with a suitable abstraction mapping α .
- ▶ Indeed, there is usually a **huge number of possibilities**, and it is important to pick good abstractions (i.e., ones that lead to informative heuristics).
- ▶ However, it is generally **not necessary to commit to a single abstraction**.

Combining Multiple Abstractions

Maximizing several abstractions:

- ▶ Each abstraction mapping gives rise to an admissible heuristic.
- ▶ By computing the **maximum** of several admissible heuristics, we obtain another admissible heuristic which **dominates** the component heuristics.
- ▶ Thus, we can always compute several abstractions and maximize over the individual abstract goal distances.

Adding several abstractions:

- ▶ In some cases, we can even compute the **sum** of individual estimates and still stay admissible.
- ▶ Summation often leads to **much higher estimates** than maximization, so it is important to understand **under which conditions** summation of heuristics is **admissible**.

Maximizing Several Abstractions: Example

Example (15-Puzzle)

- ▶ mapping to tiles 1–7 was arbitrary
 $\rightsquigarrow$ can use **any subset** of tiles
- ▶ with the same amount of memory required for the tables for the mapping to tiles 1–7, we could store the tables for **nine different abstractions** to six tiles and the blank
- ▶ use **maximum** of individual estimates

Adding Several Abstractions: Example

9	2	12	6
5	7	14	13
3	4	1	11
15	10	8	

9	2	12	6
5	7	14	13
3	4	1	11
15	10	8	

- ▶ **1st abstraction:** ignore precise location of 8–15
- ▶ **2nd abstraction:** ignore precise location of 1–7
- ~> Is the **sum** of the abstraction heuristics **admissible**?

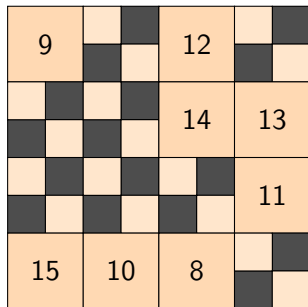
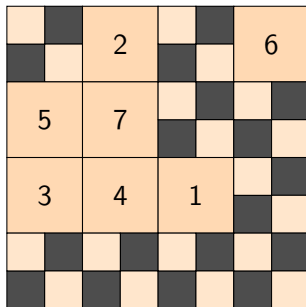
Adding Several Abstractions: Example

	2		6
5	7		
3	4	1	

9		12	
		14	13
			11
15	10	8	

- ▶ **1st abstraction:** ignore precise location of 8–15
- ▶ **2nd abstraction:** ignore precise location of 1–7
- ↪ The **sum** of the abstraction heuristics is **not admissible**.

Adding Several Abstractions: Example



- ▶ **1st abstraction:** ignore precise location of 8–15 and blank
- ▶ **2nd abstraction:** ignore precise location of 1–7 and blank
- ~> The **sum** of the abstraction heuristics is **admissible**.

E3.4 Outlook

Our Plan for the Next Lectures

In the following, we take a deeper look at abstractions and their use for admissible heuristics.

In the next two chapters, we **formally introduce** abstractions and abstraction heuristics and study some of their most important properties.

Afterwards, we discuss some particular classes of abstraction heuristics in detail, namely

- ▶ **pattern database heuristics,**
- ▶ **merge-and-shrink abstractions** and
- ▶ **Cartesian abstractions.**

E3.5 Summary

Summary

- ▶ **Abstraction** is one of the principled ways of deriving heuristics for planning tasks and transition systems in general.
- ▶ The key idea is to map states to a smaller **abstract transition system** $\mathcal{T}^\alpha$ by means of an **abstraction function** α .
- ▶ **Goal distances in $\mathcal{T}^\alpha$** are then used as **admissible estimates** for goal distances in the original transition system.
- ▶ To be **practical**, we must be able to **compute abstraction functions** and **determine abstract goal distances efficiently**.
- ▶ Often, **multiple abstractions** are used.
They can always be **maximized** admissibly.
- ▶ **Adding** abstraction heuristics is not always admissible.
When it is, it leads to a stronger heuristic than maximizing.