

Planning and Optimization

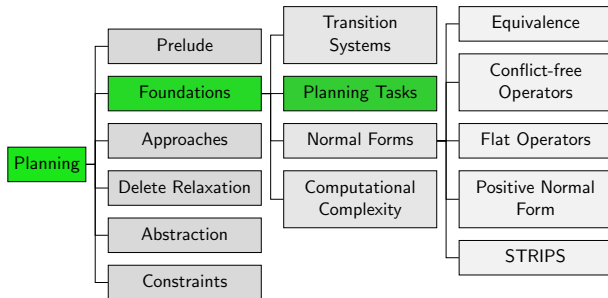
E2. Invariants and Mutexes

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Content of the Course



Invariants

Invariants

- When we as humans reason about planning tasks, we implicitly make use of “obvious” properties of these tasks.
 - **Example:** we are never in two places at the same time
- We can represent such properties as logical formulas φ that are **true in all reachable states**.
 - **Example:** $\varphi = \neg(at\text{-}uni \wedge at\text{-}home)$
- Such formulas are called **invariants** of the task.

Invariants: Definition

Definition (Invariant)

An **invariant** of a planning task Π with state variables V is a **logical formula** φ over V such that $s \models \varphi$ for **all reachable states** s of Π .

Computing Invariants

Computing Invariants

How does an **automated** planner come up with invariants?

- Theoretically, testing if a formula φ is an invariant is **as hard as planning** itself.
 $\rightsquigarrow$ **proof idea**: a planning task is **unsolvable** iff the negation of its goal is an invariant
- Still, many practical invariant synthesis algorithms exist.
- To remain efficient (= polynomial-time), these algorithms only compute a **subset** of all useful invariants.
 $\rightsquigarrow$ **sound**, but not **complete**
- Empirically, they tend to at least find the “obvious” invariants of a planning task.

Invariant Synthesis Algorithms

Most algorithms for generating invariants are based on the **generate-test-repair** approach:

- **Generate:** Suggest some invariant candidates, e.g., by enumerating all possible formulas φ of a certain size.
- **Test:** Try to prove that φ is indeed an invariant. Usually done **inductively**:
 - ① Test that **initial state** satisfies φ .
 - ② Test that if φ is true in the current state, it remains true after applying a single operator.
- **Repair:** If invariant test fails, replace candidate φ by a **weaker** formula, ideally exploiting **why** the proof failed.

Invariant Synthesis: References

We will not cover invariant synthesis algorithms in this course.

Literature on invariant synthesis:

- DISCOPLAN (Gerevini & Schubert, 1998)
- TIM (Fox & Long, 1998)
- Edelkamp & Helmert's algorithm (1999)
- Bonet & Geffner's algorithm (2001)
- Rintanen's algorithm (2008)
- Rintanen's algorithm for schematic invariants (2017)

Exploiting Invariants

Invariants have many uses in planning:

- Regression search (C3–C4):
Prune subgoals that violate (are inconsistent with) invariants.
- Planning as satisfiability (C5–C6):
Add invariants to a SAT encoding of a planning task to get tighter constraints.
- Proving unsolvability:
If φ is an invariant such that $\varphi \wedge \gamma$ is unsatisfiable, the planning task with goal γ is unsolvable.
- Finite-Domain Reformulation:
Derive a more compact FDR representation (equivalent, but with fewer states) from a given propositional planning task.

We now discuss the last point because it connects to our discussion of propositional vs. FDR planning tasks.

Mutexes

Reminder: Blocks World (Propositional Variables)

Example

$$s(A\text{-on-}B) = \mathbf{F}$$

$$s(A\text{-on-}C) = \mathbf{F}$$

$$s(A\text{-on-table}) = \mathbf{T}$$

$$s(B\text{-on-}A) = \mathbf{T}$$

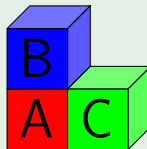
$$s(B\text{-on-}C) = \mathbf{F}$$

$$s(B\text{-on-table}) = \mathbf{F}$$

$$s(C\text{-on-}A) = \mathbf{F}$$

$$s(C\text{-on-}B) = \mathbf{F}$$

$$s(C\text{-on-table}) = \mathbf{T}$$



$\rightsquigarrow 2^9 = 512$ states

Reminder: Blocks World (Finite-Domain Variables)

Example

Use three finite-domain state variables:

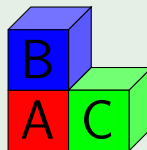
- $below-a: \{b, c, table\}$
- $below-b: \{a, c, table\}$
- $below-c: \{a, b, table\}$

$$s(below-a) = table$$

$$s(below-b) = a$$

$$s(below-c) = table$$

$$\rightsquigarrow 3^3 = 27 \text{ states}$$



Task Reformulation

- Common modeling languages (like PDDL) often give us **propositional** tasks.
- More compact FDR tasks are often desirable.
- Can we do an **automatic reformulation**?

Mutexes

Invariants that take the form of **binary clauses** are called **mutexes** because they express that certain variable assignments cannot be simultaneously true (are **mutually exclusive**).

Example (Blocks World)

The invariant $\neg A\text{-on-}B \vee \neg A\text{-on-}C$ states that $A\text{-on-}B$ and $A\text{-on-}C$ are mutex.

We say that a **set of literals** is a **mutex group** if every subset of two literals is a mutex.

Example (Blocks World)

$\{A\text{-on-}B, A\text{-on-}C, A\text{-on-table}\}$ is a mutex group.

Encoding Mutex Groups as Finite-Domain Variables

Let $G = \{\ell_1, \dots, \ell_n\}$ be a mutex group over n different propositional state variables $V_G = \{v_1, \dots, v_n\}$.

Then a single **finite-domain** state variable v_G with $\text{dom}(v_G) = \{\ell_1, \dots, \ell_n, \text{none}\}$ can replace the n variables V_G :

- $s(v_G) = \ell_i$ represents situations where (exactly) ℓ_i is true
- $s(v_G) = \text{none}$ represents situations where all ℓ_i are false

Note: We can omit the “none” value if $\ell_1 \vee \dots \vee \ell_n$ is an invariant.

Mutex Covers

Definition (Mutex Cover)

A **mutex cover** for a propositional planning task Π is a set of mutex groups $\{G_1, \dots, G_n\}$ where each variable of Π occurs in exactly one group G_i .

A mutex cover is **positive** if all literals in all groups are positive.

Note: always exists (use trivial group $\{v\}$ if v otherwise uncovered)

Positive Mutex Covers

In the following, we stick to **positive** mutex covers for simplicity.

If we have $\neg v$ in G for some group G in the cover, we can reformulate the task to use an “opposite” variable $\hat{v}$ instead, as in the conversion to positive normal form ([Chapter B5](#)).

Reformulation

Mutex-Based Reformulation of Propositional Tasks

Given a **conflict-free** propositional planning task Π with positive mutex cover $\{G_1, \dots, G_n\}$:

- In all **conditions** where variable $v \in G_i$ occurs, replace v with $v_{G_i} = v$.
- In all effects e where variable $v \in G_i$ occurs,
 - Replace all **atomic add effects** v with $v_{G_i} := v$
 - Replace all **atomic delete effects** $\neg v$ with $(v_{G_i} = v \wedge \neg \bigvee_{v' \in G_i \setminus \{v\}} \text{effcond}(v', e)) \triangleright v_{G_i} := \text{none}$

This results in an FDR planning task Π' that is equivalent to Π (without proof).

Note: the conditional effects encoding delete effects can often be simplified away to an unconditional or empty effect.

And Back?

- It can also be useful to reformulate an **FDR task** into a **propositional task**.
- For example, we might want positive normal form, which requires a propositional task.
- Key idea: each variable/value combination $v = d$ becomes a separate propositional state variable $\langle v, d \rangle$

Converting FDR Tasks into Propositional Tasks

Definition (Induced Propositional Planning Task)

Let $\Pi = \langle V, I, O, \gamma \rangle$ be a **conflict-free** FDR planning task.

The **induced propositional planning task** Π'

is the propositional planning task $\Pi' = \langle V', I', O', \gamma' \rangle$, where

- $V' = \{ \langle v, d \rangle \mid v \in V, d \in \text{dom}(v) \}$
- $I'(\langle v, d \rangle) = \mathbf{T}$ iff $I(v) = d$
- O' and γ' are obtained from O and γ by
 - replacing each atomic formula $v = d$ by the proposition $\langle v, d \rangle$
 - replacing each atomic effect $v := d$ by the effect $\langle v, d \rangle \wedge \bigwedge_{d' \in \text{dom}(v) \setminus \{d\}} \neg \langle v, d' \rangle$.

Notes:

- Again, simplifications are often possible to avoid introducing so many delete effects.
- SAS⁺ tasks induce STRIPS tasks.

Summary

Summary (1)

- **Invariants** are common properties of all reachable states, expressed as formulas.
- A number of algorithms for **computing invariants** exist.
- These algorithms will not find **all useful invariants** (which is too hard), but try to find some useful subset with reasonable (polynomial) computational effort.

Summary (2)

- **Mutexes** are invariants that express that certain literals are mutually exclusive.
- **Mutex covers** provide a way to convert a set of propositional state variables into a potentially much smaller set of finite-domain state variables.
- Using mutex covers, we can **reformulate propositional tasks** as more compact FDR tasks.
- Conversely, we can **reformulate FDR tasks** as propositional tasks by introducing propositions for each variable/value pair.