

Planning and Optimization

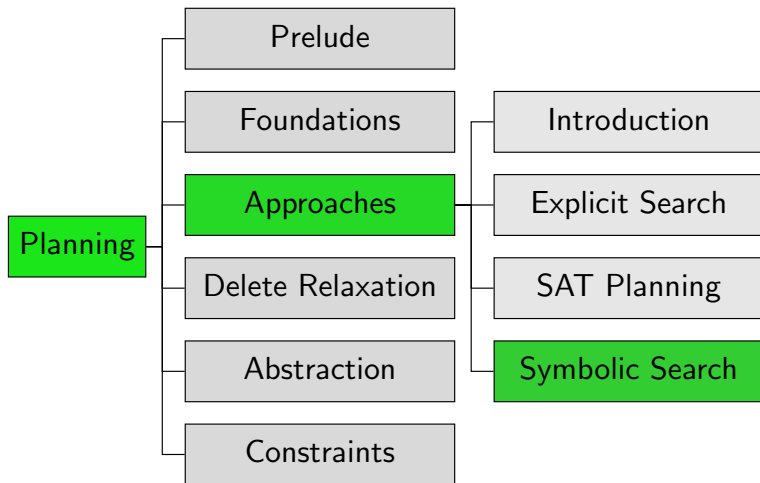
C8. Symbolic Search: Full Algorithm

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Content of the Course



Devising a Symbolic Search Algorithm

- We now put the pieces together to build a symbolic search algorithm for propositional planning tasks.
- use BDDs as a **black box** data structure:
 - care about provided operations and their time complexity
 - do not care about their internal implementation
- Efficient implementations are available as libraries, e.g.:
 - **CUDD**, a high-performance BDD library
 - **libbdd**, shipped with Ubuntu Linux

Basic BDD Operations

BDD Operations: Preliminaries

- All BDDs work on a **fixed** and **totally ordered** set of propositional variables.
- Complexity of operations given in terms of:
 - k , the number of **BDD variables**
 - $\|B\|$, the number of **nodes** in the BDD B

BDD Operations (1)

BDD operations: **logical/set atoms**

- `bdd-fullset()`: build BDD representing all assignments
 - in logic: $\top$
 - time complexity: $O(1)$
- `bdd-emptyset()`: build BDD representing $\emptyset$
 - in logic: $\perp$
 - time complexity: $O(1)$
- `bdd-atom( $v$ )`: build BDD representing $\{s \mid s(v) = \mathbf{T}\}$
 - in logic: v
 - time complexity: $O(1)$

BDD Operations (2)

BDD operations: **logical/set connectives**

- **bdd-complement**(B): build BDD representing $\overline{r(B)}$
 - in logic: $\neg\varphi$
 - time complexity: $O(\|B\|)$
- **bdd-union**(B, B'): build BDD representing $r(B) \cup r(B')$
 - in logic: $(\varphi \vee \psi)$
 - time complexity: $O(\|B\| \cdot \|B'\|)$
- **bdd-intersection**(B, B'): build BDD representing $r(B) \cap r(B')$
 - in logic: $(\varphi \wedge \psi)$
 - time complexity: $O(\|B\| \cdot \|B'\|)$

BDD Operations (3)

BDD operations: **Boolean tests**

- **bdd-includes**(B, I): return **true** iff $I \in r(B)$
 - in logic: $I \models \varphi$?
 - time complexity: $O(k)$
- **bdd-equals**(B, B'): return **true** iff $r(B) = r(B')$
 - in logic: $\varphi \equiv \psi$?
 - time complexity: $O(1)$ (due to canonical representation)

Conditioning: Formulas

The last two basic BDD operations are a bit more unusual and require some preliminary remarks.

Conditioning a variable v in a **formula** φ to **T** or **F**, written $\varphi[\mathbf{T}/v]$ or $\varphi[\mathbf{F}/v]$, means restricting v to a particular truth value:

Examples:

- $(A \wedge (B \vee \neg C))[\mathbf{T}/B] = (A \wedge (\top \vee \neg C)) \equiv A$
- $(A \wedge (B \vee \neg C))[\mathbf{F}/B] = (A \wedge (\perp \vee \neg C)) \equiv A \wedge \neg C$

Conditioning: Sets of Assignments

We can define the same operation for sets of assignments S :

$S[\mathbf{F}/v]$ and $S[\mathbf{T}/v]$ restrict S to elements with the given value for v and **remove** v from the domain of definition:

Example:

$$\blacksquare S = \{\{A \mapsto \mathbf{F}, B \mapsto \mathbf{F}, C \mapsto \mathbf{F}\}, \\ \{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{F}\}, \\ \{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$$

$$\rightsquigarrow S[\mathbf{T}/B] = \{\{A \mapsto \mathbf{T}, C \mapsto \mathbf{F}\}, \\ \{A \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$$

Forgetting

Forgetting (a.k.a. **existential abstraction**) is similar to conditioning: we allow **either** truth value for v and remove the variable.

We write this as $\exists v \varphi$ (for formulas) and $\exists v S$ (for sets).

Formally:

- $\exists v \varphi = \varphi[\mathbf{T}/v] \vee \varphi[\mathbf{F}/v]$
- $\exists v S = S[\mathbf{T}/v] \cup S[\mathbf{F}/v]$

Forgetting: Example

Examples:

$$\blacksquare S = \{\{A \mapsto \mathbf{F}, B \mapsto \mathbf{F}, C \mapsto \mathbf{F}\}, \\ \{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{F}\}, \\ \{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$$

$$\rightsquigarrow \exists B S = \{\{A \mapsto \mathbf{F}, C \mapsto \mathbf{F}\}, \\ \{A \mapsto \mathbf{T}, C \mapsto \mathbf{F}\}, \\ \{A \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$$

$$\rightsquigarrow \exists C S = \{\{A \mapsto \mathbf{F}, B \mapsto \mathbf{F}\}, \\ \{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}\}\}$$

BDD Operations (4)

BDD operations: **conditioning and forgetting**

- **bdd-condition**(B, v, t) where $t \in \{\mathbf{T}, \mathbf{F}\}$:
build BDD representing $r(B)[t/v]$
 - in logic: $\varphi[t/v]$
 - time complexity: $O(\|B\|)$
- **bdd-forget**(B, v):
build BDD representing $\exists v r(B)$
 - in logic: $\exists v \varphi \quad (= \varphi[\mathbf{T}/v] \vee \varphi[\mathbf{F}/v])$
 - time complexity: $O(\|B\|^2)$

Formulas and Singletons

Formulas to BDDs

- With the logical/set operations, we can convert propositional **formulas** φ into BDDs representing the **models** of φ .
- We denote this computation with `bdd-formula( $\varphi$ )`.
- Each individual logical connective takes **polynomial** time, but converting a full formula of length n can take $O(2^n)$ time. (How is this possible?)

Singleton BDDs

- We can convert a **single truth assignment** I into a BDD representing $\{I\}$ by computing the conjunction of all literals true in I (using `bdd-atom`, `bdd-complement` and `bdd-intersection`).
- We denote this computation with `bdd-singleton( $I$ )`.
- When done in the correct order, this takes time $O(k)$.

Renaming

Renaming

We will need to support one final operation on formulas: **renaming**.

Renaming X to Y in formula φ , written $\varphi[X \rightarrow Y]$, means **replacing** all occurrences of X by Y in φ .

We require that Y is **not present** in φ initially.

Example:

$$\blacksquare \varphi = (A \wedge (B \vee \neg C))$$

$$\rightsquigarrow \varphi[A \rightarrow D] = (D \wedge (B \vee \neg C))$$

How Hard Can That Be?

- For formulas, renaming is a **simple** (linear-time) operation.
- For a BDD B , it is equally simple ($O(\|B\|)$) when renaming between variables that are **adjacent** in the variable order.
- In general, it requires $O(\|B\|^2)$, using the equivalence $\varphi[X \rightarrow Y] \equiv \exists X(\varphi \wedge (X \leftrightarrow Y))$

Symbolic Breadth-first Search

Planning Task State Variables vs. BDD Variables

Consider propositional planning task $\langle V, I, O, \gamma \rangle$ with states S .

In symbolic planning, we have **two BDD variables** v and v' for every state variable $v \in V$ of the planning task.

- use **unprimed** variables v to describe sets of **states**:
 $\{s \in S \mid \text{some property}\}$
- use combinations of **unprimed** and **primed** variables v, v' to describe sets of **state pairs**:
 $\{\langle s, s' \rangle \mid \text{some property}\}$

Breadth-first Search with Progression and BDDs

Progression Breadth-first Search

```
def bfs-progression( $V, I, O, \gamma$ ):  
     $goal\_states := models(\gamma)$   
     $reached_0 := \{I\}$   
     $i := 0$   
    loop:  
        if  $reached_i \cap goal\_states \neq \emptyset$ :  
            return solution found  
         $reached_{i+1} := reached_i \cup apply(reached_i, O)$   
        if  $reached_{i+1} = reached_i$ :  
            return no solution exists  
         $i := i + 1$ 
```

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Use *bdd-formula*.

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Use *bdd-singleton*.

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Use *bdd-intersection*, *bdd-emptyset*, *bdd-equals*.

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def bfs-progression( $V, I, O, \gamma$ ):  
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        if  $reached_{i+1} = reached_i$ :  
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```

Use *bdd-union*.

Breadth-first Search with Progression and BDDs

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```

Use *bdd-equals*.

Breadth-first Search with Progression and BDDs

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```

How to do this?

The *apply* Function (1)

We need an operation that

- for a set of states *reached* (given as a BDD)
- and a set of operators O
- computes the set of states (as a BDD) that result from applying some operator $o \in O$ in some state $s \in \textit{reached}$.

We have seen something similar already...

Translating Operators into Formulas

Definition (Operators in Propositional Logic)

Let o be an operator and V a set of state variables.

Define $\tau_V(o) := pre(o) \wedge \bigwedge_{v \in V} (regr(v, eff(o)) \leftrightarrow v')$.

States that o is applicable and describes how

- the **new value of v** , represented by v' ,
- must relate to the **old state**, described by variables V .

The *apply* Function (2)

- The formula $\tau_V(o)$ describes all transitions $s \xrightarrow{o} s'$
 - induced by a **single** operator o
 - in terms of variables V describing s
 - and variables V' describing s' .
- The formula $\bigvee_{o \in O} \tau_V(o)$ describes state transitions by **any** operator in O .
- We can translate this formula to a BDD (over variables $V \cup V'$) with ***bdd-formula***.
- The resulting BDD is called the **transition relation** of the planning task, written as $T_V(O)$.

The *apply* Function (3)

Using the transition relation, we can compute *apply*(*reached*, *O*) as follows:

The apply function

```
def apply(reached, O):  
     $B := T_V(O)$   
     $B := \text{bdd-intersection}(B, \text{reached})$   
    for each  $v \in V$ :  
         $B := \text{bdd-forget}(B, v)$   
    for each  $v \in V$ :  
         $B := \text{bdd-rename}(B, v', v)$   
    return  $B$ 
```


The *apply* Function (3)

Using the transition relation, we can compute *apply*(*reached*, *O*) as follows:

The apply function

```
def apply(reached, O):  
    B :=  $T_v(O)$   
    B := bdd-intersection(B, reached)  
    for each  $v \in V$ :  
        B := bdd-forget(B, v)  
    for each  $v \in V$ :  
        B := bdd-rename(B,  $v'$ , v)  
    return B
```

This describes the set of **state pairs** $\langle s, s' \rangle$ where s' is a successor of s in terms of variables $V \cup V'$.

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         $B := \text{bdd-rename}(B, v', v)$   
    return  $B$ 
```

This describes the set of state pairs $\langle s, s' \rangle$ where s' is a successor of s and $s \in \textit{reached}$ in terms of variables $V \cup V'$.

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This describes the set of states s' which are successors of some state $s \in \textit{reached}$ in terms of variables V' .

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    return  $B$ 
```

Thus, *apply* indeed computes the set of successors of *reached* using operators *O*.

Discussion

Discussion

- This completes the discussion of a (basic) symbolic search algorithm for classical planning.
- We ignored the aspect of **solution extraction**. This needs some extra work, but is not a major challenge.
- In practice, some steps can be performed slightly more efficiently, but these are comparatively minor details.

Variable Orders

For good performance, we need a **good variable ordering**.

- Variables that refer to the same state variable before and after operator application (v and v') should be **neighbors** in the transition relation BDD.

Extensions

Symbolic search can be extended to...

- **regression and bidirectional search:**
this is very easy and often effective
- **uniform-cost search:**
requires some work, but not too difficult in principle
- **heuristic search:**
requires a heuristic representable as a BDD;
has not really been shown to outperform blind symbolic search

Literature (1)



Randal E. Bryant.

Graph-Based Algorithms for Boolean Function Manipulation.

IEEE Transactions on Computers 35.8, pp. 677–691, 1986.

Reduced ordered BDDs.



Kenneth L. McMillan.

Symbolic Model Checking.

PhD Thesis, 1993.

Symbolic search with BDDs.

Literature (2)



Álvaro Torralba.

Symbolic Search and Abstraction Heuristics
for Cost-Optimal Planning.

PhD Thesis, 2015.

State of the art of symbolic search planning.



David Speck, Jendrik Seipp and Álvaro Torralba.

Symbolic Search for Cost-Optimal Planning
with Expressive Model Extensions.

Journal of Artificial Intelligence Research 82,
pp. 1349–1405, 2025.

More general classes of planning tasks.

Summary

Summary

- Symbolic search operates on sets of states instead of individual states as in explicit-state search.
- State sets and transition relations can be represented as BDDs.
- Based on this, we can implement a blind breadth-first search in an efficient way.
- A good variable ordering is crucial for performance.