Planning and Optimization

C8. Symbolic Search: Full Algorithm

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October 15, 2025

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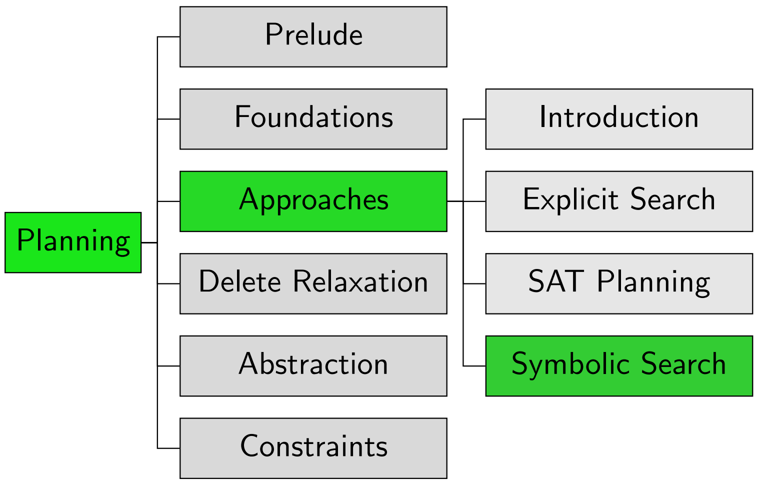
Content of the Course



Devising a Symbolic Search Algorithm

- We now put the pieces together to build a symbolic search algorithm for propositional planning tasks.
- use BDDs as a black box data structure:
  - care about provided operations and their time complexity
  - do not care about their internal implementation
- Efficient implementations are available as libraries, e.g.:
  - CUDD, a high-performance BDD library
  - libbdd, shipped with Ubuntu Linux

C8.1 Basic BDD Operations

BDD Operations: Preliminaries

- All BDDs work on a fixed and totally ordered set of propositional variables.
- Complexity of operations given in terms of:
  - k, the number of BDD variables
  - $\|B\|$, the number of nodes in the BDD B

BDD Operations (1)

BDD operations: logical/set atoms

- bdd-fullset(): build BDD representing all assignments
  - in logic: $\top$
  - time complexity: $O(1)$
- bdd-emptyset(): build BDD representing $\emptyset$
  - in logic: $\bot$
  - time complexity: $O(1)$
- bdd-atom(v): build BDD representing $\{s \mid s(v) = \mathbf{T}\}$
  - in logic: v
  - time complexity: $O(1)$

BDD Operations (2)

BDD operations: logical/set connectives

- bdd-complement(B): build BDD representing $\overline{r(B)}$
  - in logic: $\neg\varphi$
  - time complexity: $O(\|B\|)$
- bdd-union(B, B'): build BDD representing $r(B) \cup r(B')$
  - in logic: $(\varphi \vee \psi)$
  - time complexity: $O(\|B\| \cdot \|B'\|)$
- bdd-intersection(B, B'): build BDD representing $r(B) \cap r(B')$
  - in logic: $(\varphi \wedge \psi)$
  - time complexity: $O(\|B\| \cdot \|B'\|)$

BDD Operations (3)

BDD operations: Boolean tests

- bdd-includes(B, I): return **true** iff $I \in r(B)$
  - in logic: $I \models \varphi$?
  - time complexity: $O(k)$
- bdd-equals(B, B'): return **true** iff $r(B) = r(B')$
  - in logic: $\varphi \equiv \psi$?
  - time complexity: $O(1)$ (due to canonical representation)

Conditioning: Formulas

The last two basic BDD operations are a bit more unusual and require some preliminary remarks.

Conditioning a variable v in a formula φ to $\mathbf{T}$ or $\mathbf{F}$, written $\varphi[\mathbf{T}/v]$ or $\varphi[\mathbf{F}/v]$, means restricting v to a particular truth value:

Examples:

- $(A \land (B \lor \neg C))[\mathbf{T}/B] = (A \land (\top \lor \neg C)) \equiv A$
- $(A \land (B \lor \neg C))[\mathbf{F}/B] = (A \land (\bot \lor \neg C)) \equiv A \land \neg C$

Conditioning: Sets of Assignments

We can define the same operation for sets of assignments S: $S[\mathbf{F}/v]$ and $S[\mathbf{T}/v]$ restrict S to elements with the given value for v and remove v from the domain of definition:

Example:

- $S = \{\{A \mapsto \mathbf{F}, B \mapsto \mathbf{F}, C \mapsto \mathbf{F}\}, \{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{F}\}, \{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$

$\rightsquigarrow S[\mathbf{T}/B] = \{\{A \mapsto \mathbf{T}, C \mapsto \mathbf{F}\}, \{A \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$

Forgetting

Forgetting (a.k.a. existential abstraction) is similar to conditioning: we allow either truth value for v and remove the variable.

We write this as $\exists v\, \varphi$ (for formulas) and $\exists v\, S$ (for sets).

Formally:

- $\exists v\, \varphi = \varphi[\mathbf{T}/v] \lor \varphi[\mathbf{F}/v]$
- $\exists v\, S = S[\mathbf{T}/v] \cup S[\mathbf{F}/v]$

Forgetting: Example

Examples:

- $S = \{\{A \mapsto \mathbf{F}, B \mapsto \mathbf{F}, C \mapsto \mathbf{F}\},$
  $\{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{F}\},$
  $\{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$

$\rightsquigarrow$ $\exists B\, S = \{\{A \mapsto \mathbf{F}, C \mapsto \mathbf{F}\},$
$\{A \mapsto \mathbf{T}, C \mapsto \mathbf{F}\},$
$\{A \mapsto \mathbf{T}, C \mapsto \mathbf{T}\}\}$

$\rightsquigarrow$ $\exists C\, S = \{\{A \mapsto \mathbf{F}, B \mapsto \mathbf{F}\},$
$\{A \mapsto \mathbf{T}, B \mapsto \mathbf{T}\}\}$

BDD Operations (4)

BDD operations: conditioning and forgetting

- bdd-condition(B, v, t) where $t \in \{\mathbf{T}, \mathbf{F}\}$:
  build BDD representing $r(B)[t/v]$
  - in logic: $\varphi[t/v]$
  - time complexity: $O(\|B\|)$
- bdd-forget(B, v):
  build BDD representing $\exists v\, r(B)$
  - in logic: $\exists v\, \varphi \quad (= \varphi[\mathbf{T}/v] \lor \varphi[\mathbf{F}/v])$
  - time complexity: $O(\|B\|^2)$

C8.2 Formulas and Singletons

Formulas to BDDs

- With the logical/set operations, we can convert propositional formulas φ into BDDs representing the models of φ.
- We denote this computation with bdd-formula(φ).
- Each individual logical connective takes polynomial time,
  but converting a full formula of length n can take $O(2^n)$ time.
  (How is this possible?)

Singleton BDDs

- We can convert a single truth assignment I into a BDD representing $\{I\}$ by computing the conjunction of all literals true in I (using bdd-atom, bdd-complement and bdd-intersection).
- We denote this computation with bdd-singleton(I).
- When done in the correct order, this takes time $O(k)$.

C8.3 Renaming

Renaming

We will need to support one final operation on formulas: renaming.

Renaming X to Y in formula φ, written $\varphi[X \to Y]$, means replacing all occurrences of X by Y in φ.

We require that Y is not present in φ initially.

Example:

- $\varphi = (A \land (B \lor \neg C))$

$\rightsquigarrow$ $\varphi[A \to D] = (D \land (B \lor \neg C))$

How Hard Can That Be?

- For formulas, renaming is a simple (linear-time) operation.
- For a BDD B, it is equally simple ($O(\|B\|)$) when renaming between variables that are adjacent in the variable order.
- In general, it requires $O(\|B\|^2)$, using the equivalence $\varphi[X \to Y] \equiv \exists X(\varphi \land (X \leftrightarrow Y))$

C8.4 Symbolic Breadth-first Search

Planning Task State Variables vs. BDD Variables

Consider propositional planning task $\langle V, I, O, \gamma\rangle$ with states S.

In symbolic planning, we have two BDD variables v and v' for every state variable $v \in V$ of the planning task.

- use unprimed variables v to describe sets of states: $\{s \in S \mid \text{some property}\}$
- use combinations of unprimed and primed variables v, v' to describe sets of state pairs: $\{\langle s, s'\rangle \mid \text{some property}\}$

Breadth-first Search with Progression and BDDs

Progression Breadth-first Search

def bfs-progression(V, I, O, γ):
  $goal\_states := models(\gamma)$
  $reached_0 := \{I\}$
  $i := 0$
  loop:
    if $reached_i \cap goal\_states \neq \emptyset$:
      return solution found
    $reached_{i+1} := reached_i \cup apply(reached_i, O)$
    if $reached_{i+1} = reached_i$:
      return no solution exists
    $i := i + 1$

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Use *bdd-formula*.

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Use *bdd-singleton*.

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Use *bdd-intersection*, *bdd-emptyset*, *bdd-equals*.

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Use *bdd-union*.

Breadth-first Search with Progression and BDDs

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Use *bdd-equals*.

Breadth-first Search with Progression and BDDs

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How to do this?

The *apply* Function (1)

We need an operation that

- for a set of states *reached* (given as a BDD)
- and a set of operators O
- computes the set of states (as a BDD) that result from applying some operator $o \in O$ in some state $s \in reached$.

We have seen something similar already. . .

Translating Operators into Formulas

Definition (Operators in Propositional Logic)

Let o be an operator and V a set of state variables.

Define $\tau_V(o) := pre(o) \wedge \bigwedge_{v \in V}(regr(v, eff(o)) \leftrightarrow v')$.

States that o is applicable and describes how

- the new value of v, represented by v',
- must relate to the old state, described by variables V.

The *apply* Function (2)

- The formula $\tau_V(o)$ describes all transitions $s \xrightarrow{o} s'$
  - induced by a single operator o
  - in terms of variables V describing s
  - and variables V' describing s'.
- The formula $\bigvee_{o \in O} \tau_V(o)$ describes state transitions by any operator in O.
- We can translate this formula to a BDD (over variables $V \cup V'$) with *bdd-formula*.
- The resulting BDD is called the transition relation of the planning task, written as $T_V(O)$.

The *apply* Function (3)

Using the transition relation, we can compute $apply(reached, O)$ as follows:

The apply function

def apply(*reached*, O):
    $B := T_V(O)$
    $B := \textit{bdd-intersection}(B, \textit{reached})$
    for each $v \in V$:
        $B := \textit{bdd-forget}(B, v)$
    for each $v \in V$:
        $B := \textit{bdd-rename}(B, v', v)$
    return B

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This describes the set of state pairs $\langle s, s' \rangle$ where s' is a successor of s in terms of variables $V \cup V'$.

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        $B := \textit{bdd-rename}(B, v', v)$
    return B

This describes the set of state pairs $\langle s, s' \rangle$ where s' is a successor of s and $s \in \textit{reached}$ in terms of variables $V \cup V'$.

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    return B

This describes the set of states s' which are successors of some state $s \in \textit{reached}$ in terms of variables V'.

The *apply* Function (3)

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The apply function

def apply(*reached*, *O*):
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  for each $v \in V$:
    $B :=$ *bdd-rename*(B, v', v)
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This describes the set of states s' which are successors of some state $s \in$ *reached* in terms of variables V.

The *apply* Function (3)

Using the transition relation, we can compute *apply*(*reached*, *O*) as follows:

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  return B

Thus, *apply* indeed computes the set of successors of *reached* using operators O.

C8.5 Discussion

Discussion

- This completes the discussion of a (basic) symbolic search algorithm for classical planning.
- We ignored the aspect of solution extraction. This needs some extra work, but is not a major challenge.
- In practice, some steps can be performed slightly more efficiently, but these are comparatively minor details.

Variable Orders

For good performance, we need a good variable ordering.

- Variables that refer to the same state variable before and after operator application (v and v') should be neighbors in the transition relation BDD.

Extensions

Symbolic search can be extended to...

- regression and bidirectional search: this is very easy and often effective
- uniform-cost search: requires some work, but not too difficult in principle
- heuristic search: requires a heuristic representable as a BDD; has not really been shown to outperform blind symbolic search

Literature (1)

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Literature (2)

Álvaro Torralba.
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PhD Thesis, 2015.
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Journal of Artificial Intelligence Research 82, pp. 1349–1405, 2025.
More general classes of planning tasks.

C8.6 Summary

Summary

- Symbolic search operates on sets of states instead of individual states as in explicit-state search.
- State sets and transition relations can be represented as BDDs.
- Based on this, we can implement a blind breadth-first search in an efficient way.
- A good variable ordering is crucial for performance.