Planning and Optimization

C5. SAT Planning: Core Idea and Sequential Encoding

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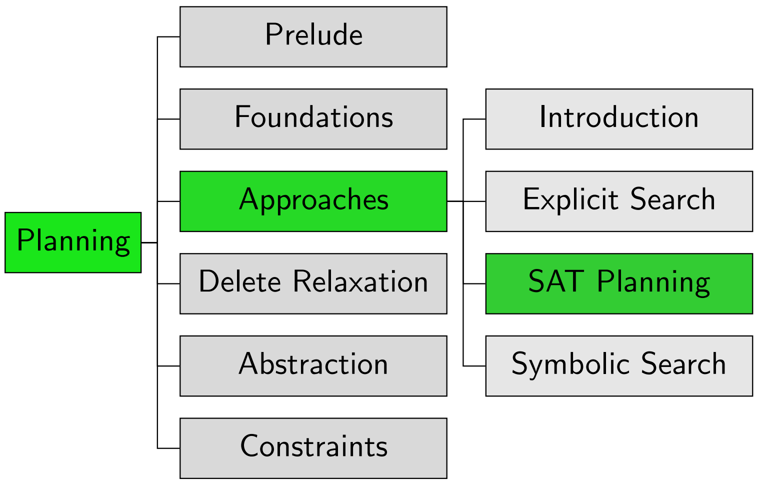
Content of the Course



C5.1 Introduction

SAT Solvers

- SAT solvers (algorithms that find satisfying assignments to CNF formulas) are one of the major success stories in solving hard combinatorial problems.
- Can we leverage them for classical planning?

$\rightsquigarrow$ SAT planning (a.k.a. planning as satisfiability)

background on SAT Solvers:
$\rightsquigarrow$ Foundations of Artificial Intelligence Course, Ch. E4–E5

Complexity Mismatch

- The SAT problem is NP-complete, while PlanEx is PSPACE-complete.

$\rightsquigarrow$ one-shot polynomial reduction from PlanEx to SAT not possible (unless NP = PSPACE)

Solution: Iterative Deepening

- We can generate a propositional formula that tests if task Π has a plan with horizon (length bound) T in time $O(\|\Pi\|^k \cdot T)$ ($\rightsquigarrow$ pseudo-polynomial reduction).
- Use as building block of algorithm that probes increasing horizons (a bit like IDA*).
- Can be efficient if there exist plans that are not excessively long.

SAT Planning: Main Loop

basic SAT planning algorithm:

SAT Planning

def satplan(Π):
    for $T \in \{0, 1, 2, \dots\}$:
        φ := build_sat_formula(Π, T)
        I = sat_solver(φ) ▷ returns a model or **none**
        if I **is not none**:
            return extract_plan(Π, T, I)

Termination criterion for unsolvable tasks?

C5.2 Formula Overview

SAT Formula: CNF?

- SAT solvers require conjunctive normal form (CNF), i.e., formulas expressed as collection of clauses.
- We will make sure that our SAT formulas are in CNF when our input is a STRIPS task.
- We do allow fully general propositional tasks, but then the formula may need additional conversion to CNF.

SAT Formula: Variables

- given propositional planning task $\Pi = \langle V, I, O, \gamma \rangle$
- given horizon $T \in \mathbb{N}_0$

Variables of the SAT Formula

- propositional variables v^i for all $v \in V$, $0 \leq i \leq T$ encode state after i steps of the plan
- propositional variables o^i for all $o \in O$, $1 \leq i \leq T$ encode operator(s) applied in i-th step of the plan

Formulas with Time Steps

Definition (Time-Stamped Formulas)

Let φ be a propositional logic formula over the variables V.
Let $0 \leq i \leq T$.

We write φ^i for the formula obtained from φ by replacing each $v \in V$ with v^i.

Example: $((a \wedge b) \vee \neg c)^3 = (a^3 \wedge b^3) \vee \neg c^3$

SAT Formula: Motivation

We want to express a formula whose models are exactly the plans/traces with T steps.

For this, the formula must express four things:

- The variables v^0 ($v \in V$) define the initial state.
- The variables v^T ($v \in V$) define a goal state.
- We select exactly one operator variable o^i ($o \in O$) for each time step $1 \leq i \leq T$.
- If we select o^i, then variables v^{i-1} and v^i ($v \in V$) describe a state transition from the $(i-1)$-th state of the plan to the i-th state of the plan (that uses operator o).

The final formula is the conjunction of all these parts.

C5.3 Initial State, Goal, Operator Selection

SAT Formula: Initial State

SAT Formula: Initial State

initial state clauses:

- v^0 for all $v \in V$ with $I(v) = \mathbf{T}$
- $\neg v^0$ for all $v \in V$ with $I(v) = \mathbf{F}$

SAT Formula: Goal

SAT Formula: Goal

goal clauses:

- γ^T

For STRIPS, this is a conjunction of unit clauses.
For general goals, this may not be in clause form.

SAT Formula: Operator Selection

Let $O = \{o_1, \dots, o_n\}$.

SAT Formula: Operator Selection

operator selection clauses:

- $o_1^i \vee \dots \vee o_n^i$ for all $1 \leq i \leq T$

operator exclusion clauses:

- $\neg o_j^i \vee \neg o_k^i$ for all $1 \leq i \leq T$, $1 \leq j < k \leq n$

C5.4 Transitions

SAT Formula: Transitions

We now get to the interesting/challenging bit:
encoding the transitions.

Key observations: if we apply operator o at time i,

- its precondition must be satisfied at time $i-1$:
  $o^i \rightarrow pre(o)^{i-1}$
- variable v is true at time i iff its regression is true at $i-1$:
  $o^i \rightarrow (v^i \leftrightarrow regr(v, eff(o))^{i-1})$

Question: Why $regr(v, eff(o))$, not $regr(v, o)$?

Simplifications and Abbreviations

- Let us pick the last formula apart to understand it better (and also get a CNF representation along the way).
- Let us call the formula τ ("transition"):
  $\tau = o^i \rightarrow (v^i \leftrightarrow regr(v, eff(o))^{i-1})$.
- First, some abbreviations:
  - Let $e = eff(o)$.
  - Let $\rho = regr(v, e)$ ("regression").
    We have $\rho = effcond(v, e) \vee (v \wedge \neg effcond(\neg v, e))$.
  - Let $\alpha = effcond(v, e)$ ("added").
  - Let $\delta = effcond(\neg v, e)$ ("deleted").

$\rightsquigarrow$ $\tau = o^i \rightarrow (v^i \leftrightarrow \rho^{i-1})$ with $\rho = \alpha \vee (v \wedge \neg\delta)$

Picking it Apart (1)

Reminder: $\tau = o^i \rightarrow (v^i \leftrightarrow \rho^{i-1})$ with $\rho = \alpha \vee (v \wedge \neg\delta)$

$$\begin{aligned}\tau &= o^i \rightarrow (v^i \leftrightarrow \rho^{i-1}) \\ &\equiv o^i \rightarrow ((v^i \rightarrow \rho^{i-1}) \wedge (\rho^{i-1} \rightarrow v^i)) \\ &\equiv \underbrace{(o^i \rightarrow (v^i \rightarrow \rho^{i-1}))}_{\tau_1} \wedge \underbrace{(o^i \rightarrow (\rho^{i-1} \rightarrow v^i))}_{\tau_2}\end{aligned}$$

$\rightsquigarrow$ consider this two separate constraints τ_1 and τ_2

Picking it Apart (2)

Reminder: $\tau_1 = o^i \rightarrow (v^i \rightarrow \rho^{i-1})$ with $\rho = \alpha \vee (v \wedge \neg\delta)$

$$\begin{aligned}\tau_1 &= o^i \rightarrow (v^i \rightarrow \rho^{i-1}) \\ &\equiv o^i \rightarrow (\neg\rho^{i-1} \rightarrow \neg v^i) \\ &\equiv (o^i \wedge \neg\rho^{i-1}) \rightarrow \neg v^i \\ &\equiv (o^i \wedge \neg(\alpha^{i-1} \vee (v^{i-1} \wedge \neg\delta^{i-1}))) \rightarrow \neg v^i \\ &\equiv (o^i \wedge (\neg\alpha^{i-1} \wedge (\neg v^{i-1} \vee \delta^{i-1}))) \rightarrow \neg v^i \\ &\equiv \underbrace{((o^i \wedge \neg\alpha^{i-1} \wedge \neg v^{i-1}) \rightarrow \neg v^i)}_{\tau_{11}} \wedge \underbrace{((o^i \wedge \neg\alpha^{i-1} \wedge \delta^{i-1}) \rightarrow \neg v^i)}_{\tau_{12}}\end{aligned}$$

$\rightsquigarrow$ consider this two separate constraints τ_{11} and τ_{12}

Interpreting the Constraints (1)

Can we give an intuitive description of τ_{11} and τ_{12}?
$\rightsquigarrow$ Yes!

- $\tau_{11} = (o^i \wedge \neg\alpha^{i-1} \wedge \neg v^{i-1}) \rightarrow \neg v^i$
  "When applying o, if v is false and o does not add it, it remains false."
  - called negative frame clause
  - in clause form: $\neg o^i \vee \alpha^{i-1} \vee v^{i-1} \vee \neg v^i$
- $\tau_{12} = (o^i \wedge \neg\alpha^{i-1} \wedge \delta^{i-1}) \rightarrow \neg v^i$
  "When applying o, if o deletes v and does not add it, it is false afterwards." (Note the add-after-delete semantics.)
  - called negative effect clause
  - in clause form: $\neg o^i \vee \alpha^{i-1} \vee \neg\delta^{i-1} \vee \neg v^i$

For STRIPS tasks, these are indeed clauses. (And in general?)

Picking it Apart (3)

Almost done!

Reminder: $\tau_2 = o^i \rightarrow (\rho^{i-1} \rightarrow v^i)$ with $\rho = \alpha \vee (v \wedge \neg\delta)$

$$\begin{aligned}\tau_2 &= o^i \rightarrow (\rho^{i-1} \rightarrow v^i) \\ &\equiv (o^i \wedge \rho^{i-1}) \rightarrow v^i \\ &\equiv (o^i \wedge (\alpha^{i-1} \vee (v^{i-1} \wedge \neg\delta^{i-1}))) \rightarrow v^i \\ &\equiv \underbrace{((o^i \wedge \alpha^{i-1}) \rightarrow v^i)}_{\tau_{21}} \wedge \underbrace{((o^i \wedge v^{i-1} \wedge \neg\delta^{i-1}) \rightarrow v^i)}_{\tau_{22}}\end{aligned}$$

$\rightsquigarrow$ consider this two separate constraints τ_{21} and τ_{22}

Interpreting the Constraints (2)

How about an intuitive description of τ_{21} and τ_{22}?

- $\tau_{21} = (o^i \land \alpha^{i-1}) \to v^i$
  "When applying o, if o adds v, it is true afterwards."
  - called positive effect clause
  - in clause form: $\neg o^i \lor \neg\alpha^{i-1} \lor v^i$
- $\tau_{22} = (o^i \land v^{i-1} \land \neg\delta^{i-1}) \to v^i$
  "When applying o, if v is true and o does not delete it, it remains true."
  - called positive frame clause
  - in clause form: $\neg o^i \lor \neg v^{i-1} \lor \delta^{i-1} \lor v^i$

For STRIPS tasks, these are indeed clauses. (But not in general.)

SAT Formula: Transitions

SAT Formula: Transitions

precondition clauses:

- $\neg o^i \lor pre(o)^{i-1}$ for all $1 \le i \le T$, $o \in O$

positive and negative effect clauses:

- $\neg o^i \lor \neg\alpha^{i-1} \lor v^i$ for all $1 \le i \le T$, $o \in O$, $v \in V$
- $\neg o^i \lor \alpha^{i-1} \lor \neg\delta^{i-1} \lor \neg v^i$ for all $1 \le i \le T$, $o \in O$, $v \in V$

positive and negative frame clauses:

- $\neg o^i \lor \neg v^{i-1} \lor \delta^{i-1} \lor v^i$ for all $1 \le i \le T$, $o \in O$, $v \in V$
- $\neg o^i \lor \alpha^{i-1} \lor v^{i-1} \lor \neg v^i$ for all $1 \le i \le T$, $o \in O$, $v \in V$

where $\alpha = effcond(v, eff(o))$, $\delta = effcond(\neg v, eff(o))$.

For STRIPS, all except the precondition clauses are in clause form.

The precondition clauses are easily convertible to CNF (one clause $\neg o^i \lor v^{i-1}$ for each precondition atom v of o).

C5.5 Summary

Summary: Sequential SAT Encoding (1)

Sequential SAT Encoding (1)

initial state clauses:

- v^0 for all $v \in V$ with $I(v) = \mathbf{T}$
- $\neg v^0$ for all $v \in V$ with $I(v) = \mathbf{F}$

goal clauses:

- γ^T

operator selection clauses:

- $o_1^i \lor \dots \lor o_n^i$ for all $1 \le i \le T$

operator exclusion clauses:

- $\neg o_j^i \lor \neg o_k^i$ for all $1 \le i \le T$, $1 \le j < k \le n$

Summary: Sequential SAT Encoding (2)

Sequential SAT Encoding (2)

precondition clauses:

- $\neg o^i \lor pre(o)^{i-1}$ for all $1 \leq i \leq T$, $o \in O$

positive and negative effect clauses:

- $\neg o^i \lor \neg \alpha^{i-1} \lor v^i$ for all $1 \leq i \leq T$, $o \in O$, $v \in V$
- $\neg o^i \lor \alpha^{i-1} \lor \neg \delta^{i-1} \lor \neg v^i$ for all $1 \leq i \leq T$, $o \in O$, $v \in V$

positive and negative frame clauses:

- $\neg o^i \lor \neg v^{i-1} \lor \delta^{i-1} \lor v^i$ for all $1 \leq i \leq T$, $o \in O$, $v \in V$
- $\neg o^i \lor \alpha^{i-1} \lor v^{i-1} \lor \neg v^i$ for all $1 \leq i \leq T$, $o \in O$, $v \in V$

where $\alpha = \mathit{effcond}(v, \mathit{eff}(o))$, $\delta = \mathit{effcond}(\neg v, \mathit{eff}(o))$.

Summary

- SAT planning (planning as satisfiability) expresses a sequence of bounded-horizon planning tasks as SAT formulas.
- Plans can be extracted from satisfying assignments; unsolvable tasks are challenging for the algorithm.
- For each time step, there are propositions encoding which state variables are true and which operators are applied.
- We describe a basic sequential encoding where one operator is applied at every time step.
- The encoding produces a CNF formula for STRIPS tasks.
- The encoding follows naturally (with some work) from using regression to link state variables in adjacent time steps.