

# Planning and Optimization

## B7. Computational Complexity of Planning: Results

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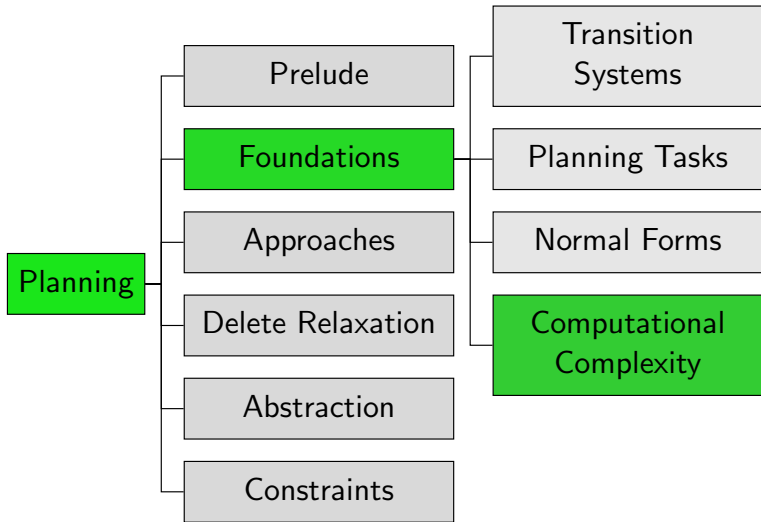
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# Content of the Course



## B7.1 (Bounded-Cost) Plan Existence

# Decision Problems for Planning

## Definition (Plan Existence)

**Plan existence (PLANEX)** is the following decision problem:

GIVEN:        planning task  $\Pi$

QUESTION:   Is there a plan for  $\Pi$ ?

$\rightsquigarrow$  decision problem analogue of **satisficing planning**

## Definition (Bounded-Cost Plan Existence)

**Bounded-cost plan existence (BCPLANEX)**

is the following decision problem:

GIVEN:        planning task  $\Pi$ , cost bound  $K \in \mathbb{N}_0$

QUESTION:   Is there a plan for  $\Pi$  with cost at most  $K$ ?

$\rightsquigarrow$  decision problem analogue of **optimal planning**

# Plan Existence vs. Bounded-Cost Plan Existence

## Theorem (Reduction from PLANEX to BCPLANEX)

$$\text{PLANEX} \leq_p \text{BCPLANEX}$$

### Proof.

Consider a planning task  $\Pi$  with state variables  $V$ .

Let  $c_{\max}$  be the maximal cost of all operators of  $\Pi$ .

Compute the number of states of  $\Pi$  as  $N = 2^{|V|}$ .

$\Pi$  is solvable iff there is solution with cost at most  $c_{\max} \cdot (N - 1)$  because a solution need not visit any state twice.

$\rightsquigarrow$  map instance  $\Pi$  of PLANEX to instance  $\langle \Pi, c_{\max} \cdot (N - 1) \rangle$  of BCPLANEX

$\rightsquigarrow$  polynomial reduction



## B7.2 PSPACE-Completeness of Planning

# Membership in PSPACE

## Theorem

$\text{BCPLANEX} \in \text{PSPACE}$

## Proof.

Show  $\text{BCPLANEX} \in \text{NPSPACE}$  and use Savitch's theorem.

Nondeterministic algorithm:

```
def plan( $\langle V, I, O, \gamma \rangle, K$ ):  
     $s := I$   
     $k := K$   
    loop forever:  
        if  $s \models \gamma$ : accept  
        guess  $o \in O$   
        if  $o$  is not applicable in  $s$ : fail  
        if  $\text{cost}(o) > k$ : fail  
         $s := s[o]$   
         $k := k - \text{cost}(o)$ 
```





# PSPACE-Hardness

Idea: generic reduction

- ▶ For an arbitrary fixed DTM  $M$  with space bound polynomial  $p$  and input  $w$ , generate propositional planning task which is solvable iff  $M$  accepts  $w$  in space  $p(|w|)$ .
- ▶ Without loss of generality, we assume  $p(n) \geq n$  for all  $n$ .

## Reduction: State Variables

Let  $M = \langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$  be the fixed DTM, and let  $p$  be its space-bound polynomial.

Given input  $w_1 \dots w_n$ , define **relevant tape positions**  $X := \{-p(n), \dots, p(n)\}$

### State Variables

- ▶  $\text{state}_q$  for all  $q \in Q$
- ▶  $\text{head}_i$  for all  $i \in X \cup \{-p(n) - 1, p(n) + 1\}$
- ▶  $\text{content}_{i,a}$  for all  $i \in X, a \in \Sigma_\square$

$\rightsquigarrow$  allows encoding a Turing machine configuration

## Reduction: Initial State

Let  $M = \langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$  be the fixed DTM,  
and let  $p$  be its space-bound polynomial.

Given input  $w_1 \dots w_n$ , define **relevant tape positions**  
 $X := \{-p(n), \dots, p(n)\}$

### Initial State

Initially true:

- ▶  $\text{state}_{q_0}$
- ▶  $\text{head}_1$
- ▶  $\text{content}_{i, w_i}$  for all  $i \in \{1, \dots, n\}$
- ▶  $\text{content}_{i, \square}$  for all  $i \in X \setminus \{1, \dots, n\}$

Initially false:

- ▶ all others

## Reduction: Operators

Let  $M = \langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$  be the fixed DTM,  
and let  $p$  be its space-bound polynomial.

Given input  $w_1 \dots w_n$ , define **relevant tape positions**  
 $X := \{-p(n), \dots, p(n)\}$

### Operators

One operator for each transition rule  $\delta(q, a) = \langle q', a', d \rangle$   
and each cell position  $i \in X$ :

- ▶ precondition:  $\text{state}_q \wedge \text{head}_i \wedge \text{content}_{i,a}$
- ▶ effect:  $\neg \text{state}_q \wedge \neg \text{head}_i \wedge \neg \text{content}_{i,a}$   
 $\wedge \text{state}_{q'} \wedge \text{head}_{i+d} \wedge \text{content}_{i,a'}$

Note that add-after-delete semantics are important here!

## Reduction: Goal

Let  $M = \langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$  be the fixed DTM,  
and let  $p$  be its space-bound polynomial.

Given input  $w_1 \dots w_n$ , define **relevant tape positions**

$$X := \{-p(n), \dots, p(n)\}$$

Goal

state $_{q_Y}$

# PSPACE-Completeness of STRIPS Plan Existence

Theorem (PSPACE-Completeness; Bylander, 1994)

$\text{PLANEX}$  and  $\text{BCPLANEX}$  are PSPACE-complete.

*This is true even if only STRIPS tasks are allowed.*

Proof.

Membership for  $\text{BCPLANEX}$  was already shown.

Hardness for  $\text{PLANEX}$  follows because we just presented a polynomial reduction from an arbitrary problem in PSPACE to  $\text{PLANEX}$ . (Note that the reduction only generates STRIPS tasks, after trivial cleanup to make them conflict-free.)

Membership for  $\text{PLANEX}$  and hardness for  $\text{BCPLANEX}$  follow from the polynomial reduction from  $\text{PLANEX}$  to  $\text{BCPLANEX}$ .  $\square$

## B7.3 More Complexity Results

# More Complexity Results

In addition to the basic complexity result presented in this chapter, there are many special cases, generalizations, variations and related problems studied in the literature:

- ▶ different **planning formalisms**
  - ▶ e.g., nondeterministic effects, partial observability, schematic operators, numerical state variables
- ▶ **syntactic restrictions** of planning tasks
  - ▶ e.g., without preconditions, without conjunctive effects, STRIPS without delete effects
- ▶ **semantic restrictions** of planning task
  - ▶ e.g., restricting variable dependencies (“causal graphs”)
- ▶ **particular planning domains**
  - ▶ e.g., Blocksworld, Logistics, FreeCell



# Complexity Results for Different Planning Formalisms

Some results for different planning formalisms:

- ▶ **nondeterministic effects:**
  - ▶ fully observable: EXP-complete (Littman, 1997)
  - ▶ unobservable: EXPSPACE-complete (Haslum & Jonsson, 1999)
  - ▶ partially observable: 2-EXP-complete (Rintanen, 2004)
- ▶ **schematic operators:**
  - ▶ usually adds one exponential level to PLANEX complexity
  - ▶ e.g., classical case EXPSPACE-complete (Erol et al., 1995)
- ▶ **numerical state variables:**
  - ▶ undecidable in most variations (Helmert, 2002)
  - ▶ decidable in restricted setting with at most two numeric state variables (Helal and Lakemeyer, 2025)

## B7.4 Summary

# Summary

- ▶ Classical planning is PSPACE-complete.
- ▶ This is true both for satisficing and optimal planning (rather, the corresponding decision problems).
- ▶ The hardness proof is a polynomial reduction that translates an arbitrary polynomial-space DTM into a STRIPS task:
  - ▶ DTM configurations are encoded by state variables.
  - ▶ Operators simulate transitions between DTM configurations.
  - ▶ The DTM accepts an input iff there is a plan for the corresponding STRIPS task.
- ▶ This implies that there is no polynomial algorithm for classical planning unless  $P = PSPACE$ .
- ▶ It also means that planning is not polynomially reducible to any problem in NP unless  $NP = PSPACE$ .