Planning and Optimization

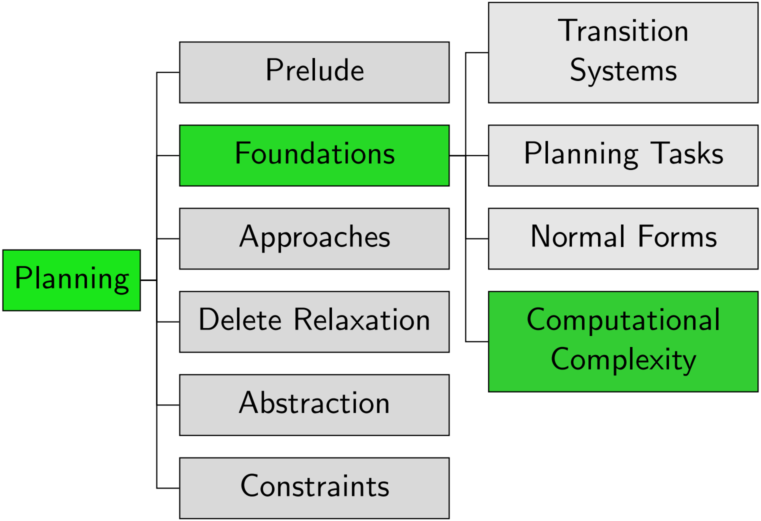
B6. Computational Complexity of Planning: Background

Malte Helmert and Gabriele Röger

Universität Basel

October 1, 2025

Content of the Course



Motivation

How Difficult is Planning?

- Using state-space search (e.g., using Dijkstra's algorithm on the transition system), planning can be solved in polynomial time in the number of states.
- However, the number of states is exponential in the number of state variables, and hence in general exponential in the size of the input to the planning algorithm.

⇝ Do non-exponential planning algorithms exist?

⇝ What is the precise computational complexity of planning?

Why Computational Complexity?

- understand the problem
- know what is not possible
- find interesting subproblems that are easier to solve
- distinguish essential features from syntactic sugar
    - Is STRIPS planning easier than general planning?

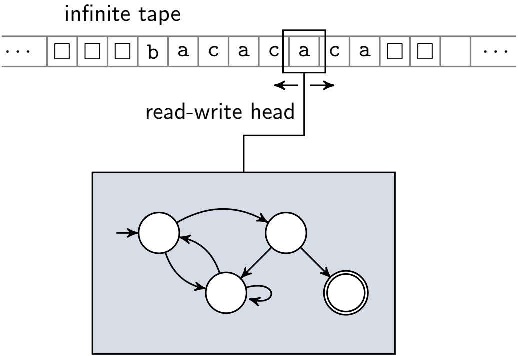
Reminder: Complexity Theory

Need to Catch Up?

- We assume knowledge of complexity theory:
  - languages and decision problems
  - Turing machines: NTMs and DTMs;
    polynomial equivalence with other models of computation
  - complexity classes: P, NP, PSPACE
  - polynomial reductions
- If you are not familiar with these topics, we recommend Chapters B11, D1–D3, D6 of the Theory of Computer Science course at `https://dmi.unibas.ch/en/studium/computer-science-informatik/lehrangebot-fs25/10948-main-lecture-theory-of-computer-science/`

Turing Machines

Turing Machines: Conceptually



Turing Machines

Definition (Nondeterministic Turing Machine)

A nondeterministic Turing machine (NTM) is a 6-tuple $\langle \Sigma, \square, Q, q_0, q_Y, \delta \rangle$ with the following components:

- input alphabet Σ and blank symbol $\square \notin \Sigma$
  - alphabets always nonempty and finite
  - tape alphabet $\Sigma_\square = \Sigma \cup \{\square\}$
- finite set Q of internal states with initial state $q_0 \in Q$ and accepting state $q_Y \in Q$
  - nonterminal states $Q' := Q \setminus \{q_Y\}$
- transition relation $\delta : (Q' \times \Sigma_\square) \to 2^{Q \times \Sigma_\square \times \{-1,+1\}}$

Deterministic Turing machine (DTM):
$|\delta(q, s)| = 1$ for all $\langle q, s \rangle \in Q' \times \Sigma_\square$

Turing Machines: Accepted Words

- Initial configuration
  - state q_0
  - input word on tape, all other tape cells contain $\Box$
  - head on first symbol of input word
- Step
  - If in state q, reading symbol s, and $\langle q', s', d\rangle \in \delta(q, s)$ then
  - the NTM can transition to state q', replacing s with s' and moving the head one cell to the left/right ($d = -1/+1$).
- Input word ($\in \Sigma^*$) is accepted if some sequence of transitions brings the NTM from the initial configuration into state q_Y.

Complexity Classes

Acceptance in Time and Space

Definition (Acceptance of a Language in Time/Space)

Let $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$.

A NTM accepts language $L \subseteq \Sigma^*$ in time f if it accepts each $w \in L$ within $f(|w|)$ steps and does not accept any $w \notin L$ (in any time).

It accepts language $L \subseteq \Sigma^*$ in space f if it accepts each $w \in L$ using at most $f(|w|)$ tape cells and does not accept any $w \notin L$.

Time and Space Complexity Classes

Definition (DTIME, NTIME, DSPACE, NSPACE)

Let $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$.

Complexity class $\mathsf{DTIME}(f)$ contains all languages accepted in time f by some DTM.

Complexity class $\mathsf{NTIME}(f)$ contains all languages accepted in time f by some NTM.

Complexity class $\mathsf{DSPACE}(f)$ contains all languages accepted in space f by some DTM.

Complexity class $\mathsf{NSPACE}(f)$ contains all languages accepted in space f by some NTM.

Polynomial Time and Space Classes

Let $\mathcal{P}$ be the set of polynomials $p : \mathbb{N}_0 \to \mathbb{N}_0$ whose coefficients are natural numbers.

Definition (P, NP, PSPACE, NPSPACE)

$$\begin{aligned}
\mathsf{P} &= \textstyle\bigcup_{p \in \mathcal{P}} \mathsf{DTIME}(p) \\
\mathsf{NP} &= \textstyle\bigcup_{p \in \mathcal{P}} \mathsf{NTIME}(p) \\
\mathsf{PSPACE} &= \textstyle\bigcup_{p \in \mathcal{P}} \mathsf{DSPACE}(p) \\
\mathsf{NPSPACE} &= \textstyle\bigcup_{p \in \mathcal{P}} \mathsf{NSPACE}(p)
\end{aligned}$$

Polynomial Complexity Class Relationships

Theorem (Complexity Class Hierarchy)

P $\subseteq$ NP $\subseteq$ PSPACE $=$ NPSPACE

Proof.

P $\subseteq$ NP and PSPACE $\subseteq$ NPSPACE are obvious because deterministic Turing machines are a special case of nondeterministic ones.

NP $\subseteq$ NPSPACE holds because a Turing machine can only visit polynomially many tape cells within polynomial time.

PSPACE $=$ NPSPACE is a special case of a classical result known as Savitch's theorem (Savitch 1970). □

Summary

Summary

- We recalled the definitions of the most important complexity classes from complexity theory:
  - P: decision problems solvable in polynomial time
  - NP: decision problems solvable in polynomial time by nondeterministic algorithms
  - PSPACE: decision problems solvable in polynomial space
  - NPSPACE: decision problems solvable in polynomial space by nondeterministic algorithms
- These classes are related by $\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} = \mathrm{NPSPACE}$.