

Planning and Optimization

B1. Transition Systems and Propositional Logic

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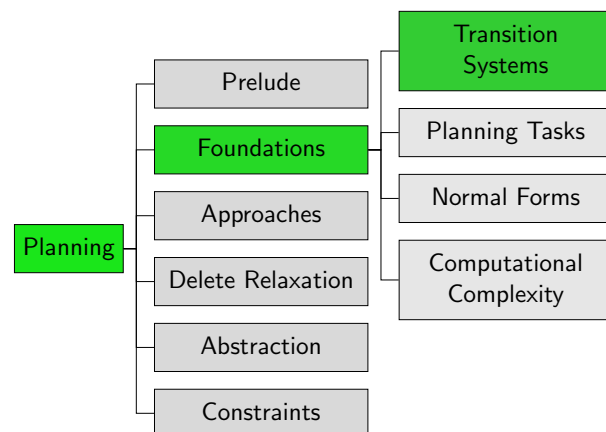
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Content of the Course



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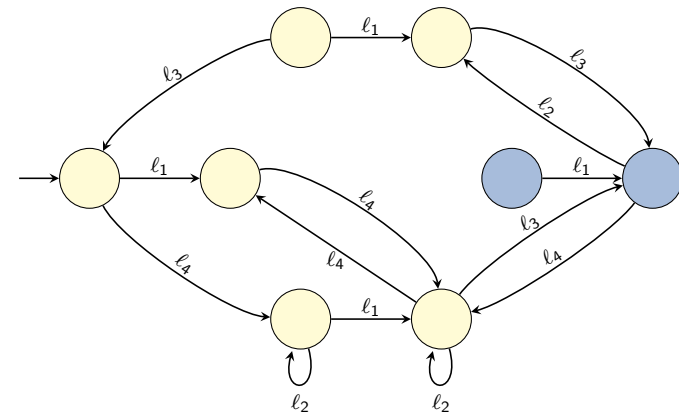
Our next steps are to formally define our problem:

- ▶ introduce a mathematical model for planning tasks:
transition systems
↪ Chapter B1
- ▶ introduce **compact representations** for planning tasks
suitable as input for planning algorithms
↪ Chapter B2

B1.1 Transition Systems

Transition System Example

Transition systems are often depicted as **directed arc-labeled graphs** with decorations to indicate the initial state and goal states.



$$c(l_1) = 1, c(l_2) = 1, c(l_3) = 5, c(l_4) = 0$$

Transition Systems

Definition (Transition System)

A **transition system** is a 6-tuple $\mathcal{T} = \langle S, L, c, T, s_0, S_* \rangle$ where

- ▶ S is a finite set of **states**,
- ▶ L is a finite set of (transition) **labels**,
- ▶ $c : L \rightarrow \mathbb{R}_0^+$ is a **label cost** function,
- ▶ $T \subseteq S \times L \times S$ is the **transition relation**,
- ▶ $s_0 \in S$ is the **initial state**, and
- ▶ $S_* \subseteq S$ is the set of **goal states**.

We say that \mathcal{T} **has the transition** $\langle s, \ell, s' \rangle$ if $\langle s, \ell, s' \rangle \in T$.

We also write this as $s \xrightarrow{\ell} s'$, or $s \rightarrow s'$ when not interested in ℓ .

Note: Transition systems are also called **state spaces**.

Deterministic Transition Systems

Definition (Deterministic Transition System)

A transition system is called **deterministic** if for all states s and all labels ℓ , there is **at most one** state s' with $s \xrightarrow{\ell} s'$.

Example: previously shown transition system

Transition System Terminology (1)

We use common terminology from graph theory:

- ▶ s' **successor** of s if $s \rightarrow s'$
- ▶ s **predecessor** of s' if $s \rightarrow s'$

Transition System Terminology (2)

We use common terminology from graph theory:

- ▶ s' **reachable** from s if there exists a sequence of transitions $s^0 \xrightarrow{\ell_1} s^1, \dots, s^{n-1} \xrightarrow{\ell_n} s^n$ s.t. $s^0 = s$ and $s^n = s'$
 - ▶ **Note:** $n = 0$ possible; then $s = s'$
 - ▶ s^0, \dots, s^n is called **(state) path** from s to s'
 - ▶ ℓ_1, \dots, ℓ_n is called **(label) path** from s to s'
 - ▶ $s^0 \xrightarrow{\ell_1} s^1, \dots, s^{n-1} \xrightarrow{\ell_n} s^n$ is called **trace** from s to s'
 - ▶ **length** of path/trace is n
 - ▶ **cost** of label path/trace is $\sum_{i=1}^n c(\ell_i)$

Transition System Terminology (3)

We use common terminology from graph theory:

- ▶ s' **reachable** (without reference state) means reachable from initial state s_0
- ▶ **solution** or **goal path** from s : path from s to some $s' \in S_*$
 - ▶ if s is omitted, $s = s_0$ is implied
- ▶ transition system **solvable** if a goal path from s_0 exists

B1.2 Example: Blocks World

Running Example: Blocks World

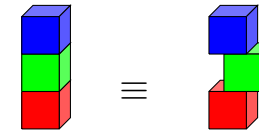
- ▶ Throughout the course, we occasionally use the **blocks world** domain as an example.
- ▶ In the blocks world, a number of different blocks are arranged on a table.
- ▶ Our job is to rearrange them according to a given goal.

Blocks World Rules (1)

Location on the table does not matter.

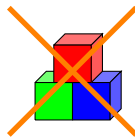


Location on a block does not matter.

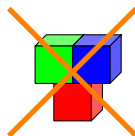


Blocks World Rules (2)

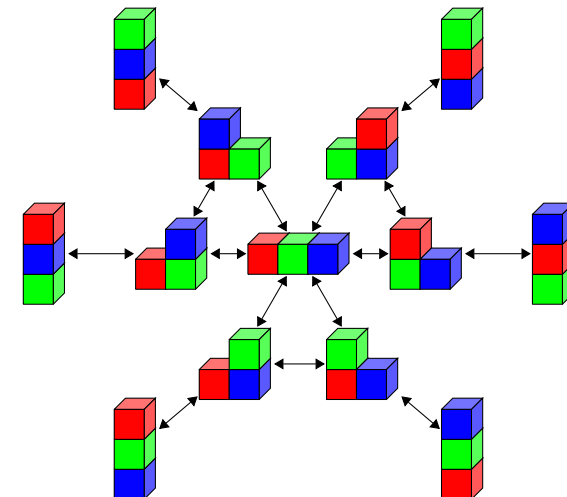
At most one block may be below a block.



At most one block may be on top of a block.



Blocks World Transition System for Three Blocks



Labels omitted for clarity. All label costs are 1. Initial/goal states not marked.

Blocks World Computational Properties

blocks	states	blocks	states
1	1	10	58941091
2	3	11	824073141
3	13	12	12470162233
4	73	13	202976401213
5	501	14	3535017524403
6	4051	15	65573803186921
7	37633	16	1290434218669921
8	394353	17	26846616451246353
9	4596553	18	588633468315403843

- ▶ Finding solutions is possible in **linear time** in the number of blocks: move everything onto the table, then construct the goal configuration.
- ▶ Finding a **shortest solution** is **NP-complete** given a compact description of the problem.

The Need for Compact Descriptions

- ▶ We see from the blocks world example that transition systems are often **far too large** to be directly used as **inputs** to planning algorithms.
- ▶ We therefore need **compact descriptions** of transition systems.
- ▶ For this purpose, we will use **propositional logic**, which allows expressing information about 2^n states as logical formulas over n **state variables**.

B1.3 Reminder: Propositional Logic

More on Propositional Logic

Need to Catch Up?

- ▶ This section is a **reminder**. We assume you are already well familiar with propositional logic.
- ▶ If this is not the case, we recommend Chapters D1–D4 of the **Discrete Mathematics in Computer Science** course: <https://dmi.unibas.ch/en/studies/computer-science/course-offer-hs24/lecture-discrete-mathematics-in-computer-science/>
 - ▶ Videos for these chapters are available on request.

Syntax of Propositional Logic

Definition (Logical Formula)

Let A be a set of **atomic propositions**.

The **logical formulas** over A are constructed by finite application of the following rules:

- ▶ \top and \perp are logical formulas (**truth** and **falsity**).
- ▶ For all $a \in A$, a is a logical formula (**atom**).
- ▶ If φ is a logical formula, then so is $\neg\varphi$ (**negation**).
- ▶ If φ and ψ are logical formulas, then so are $(\varphi \vee \psi)$ (**disjunction**) and $(\varphi \wedge \psi)$ (**conjunction**).

Syntactical Conventions for Propositional Logic

Abbreviations:

- ▶ $(\varphi \rightarrow \psi)$ is short for $(\neg\varphi \vee \psi)$ (**implication**)
- ▶ $(\varphi \leftrightarrow \psi)$ is short for $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$ (**equijunction**)
- ▶ parentheses omitted when not necessary:
 - ▶ (\neg) binds more tightly than binary connectives
 - ▶ (\wedge) binds more tightly than (\vee) , which binds more tightly than (\rightarrow) , which binds more tightly than (\leftrightarrow)

Semantics of Propositional Logic

Definition (Interpretation, Model)

An **interpretation** of propositions A is a function $I : A \rightarrow \{\mathbf{T}, \mathbf{F}\}$.

Define the notation $I \models \varphi$ (I **satisfies** φ ; I is a **model** of φ ; φ is **true** under I) for interpretations I and formulas φ by

- ▶ $I \models \top$
- ▶ $I \not\models \perp$
- ▶ $I \models a$ iff $I(a) = \mathbf{T}$ (for all $a \in A$)
- ▶ $I \models \neg\varphi$ iff $I \not\models \varphi$
- ▶ $I \models (\varphi \vee \psi)$ iff $(I \models \varphi \text{ or } I \models \psi)$
- ▶ $I \models (\varphi \wedge \psi)$ iff $(I \models \varphi \text{ and } I \models \psi)$

Note: Interpretations are also called **valuations** or **truth assignments**.

Propositional Logic Terminology (1)

- ▶ A logical formula φ is **satisfiable** if there is at least one interpretation I such that $I \models \varphi$.
- ▶ Otherwise it is **unsatisfiable**.
- ▶ A logical formula φ is **valid** or a **tautology** if $I \models \varphi$ for all interpretations I .
- ▶ A logical formula ψ is a **logical consequence** of a logical formula φ , written $\varphi \models \psi$, if $I \models \psi$ for all interpretations I with $I \models \varphi$.
- ▶ Two logical formulas φ and ψ are **logically equivalent**, written $\varphi \equiv \psi$, if $\varphi \models \psi$ and $\psi \models \varphi$.

Question: How to phrase these in terms of **models**?

Propositional Logic Terminology (2)

- ▶ A logical formula that is a proposition a or a negated proposition $\neg a$ for some atomic proposition $a \in A$ is a **literal**.
- ▶ A formula that is a disjunction of literals is a **clause**. This includes **unit clauses** ℓ consisting of a single literal and the **empty clause** \perp consisting of zero literals.
- ▶ A formula that is a conjunction of literals is a **monomial**. This includes **unit monomials** ℓ consisting of a single literal and the **empty monomial** \top consisting of zero literals.

Normal forms:

- ▶ negation normal form (NNF)
- ▶ conjunctive normal form (CNF)
- ▶ disjunctive normal form (DNF)

B1.4 Summary

Summary

- ▶ **Transition systems** are (typically huge) directed graphs that encode how the state of the world can change.
- ▶ **Propositional logic** allows us to compactly describe complex information about large sets of interpretations as **logical formulas**.