Discrete Mathematics in Computer Science

D6. Advanced Concepts in Predicate Logic and Outlook

Malte Helmert, Gabriele Röger

University of Basel

December 15, 2025

D6.1 Free and Bound Variables

Free and Bound Variables: Motivation

Question:

- Consider a signature with variable symbols $\{x_1, x_2, x_3, \dots\}$ and an interpretation $\mathcal{I}$.
- Which parts of the definition of α are relevant to decide whether $\mathcal{I}, \alpha \models (\forall x_4 (\mathsf{R}(x_4, x_2) \lor (\mathsf{f}(x_3) = x_4)) \lor \exists x_3 \mathsf{S}(x_3, x_2))$?
- $\alpha(x_1)$, $\alpha(x_5)$, $\alpha(x_6)$, $\alpha(x_7)$, ... are irrelevant since those variable symbols occur in no formula.
- $\alpha(x_4)$ also is irrelevant: the variable occurs in the formula, but all occurrences are bound by a surrounding quantifier.
- $\rightsquigarrow$ only assignments for free variables x_2 and x_3 relevant

German: gebundene und freie Variablen

Variables of a Term

Definition (Variables of a Term)

Let t be a term. The set of variables that occur in t, written as $var(t)$, is defined as follows:

- $var(x) = \{x\}$
  for variable symbols x
- $var(\mathrm{c}) = \emptyset$
  for constant symbols c
- $var(\mathrm{f}(t_1, \dots, t_k)) = var(t_1) \cup \dots \cup var(t_k)$
  for function terms

terminology: A term t with $var(t) = \emptyset$ is called ground term.
German: Grundterm

example: $var(\mathrm{product}(x, \mathrm{sum}(\mathrm{k}, y))) =$

Free and Bound Variables of a Formula

Definition (Free Variables)

Let φ be a predicate logic formula. The set of free variables of φ, written as $\mathit{free}(\varphi)$, is defined as follows:

- $\mathit{free}(\mathsf{P}(t_1, \dots, t_k)) = \mathit{var}(t_1) \cup \dots \cup \mathit{var}(t_k)$
- $\mathit{free}((t_1 = t_2)) = \mathit{var}(t_1) \cup \mathit{var}(t_2)$
- $\mathit{free}(\neg\varphi) = \mathit{free}(\varphi)$
- $\mathit{free}((\varphi \land \psi)) = \mathit{free}((\varphi \lor \psi)) = \mathit{free}(\varphi) \cup \mathit{free}(\psi)$
- $\mathit{free}(\forall x\, \varphi) = \mathit{free}(\exists x\, \varphi) = \mathit{free}(\varphi) \setminus \{x\}$

Example: $\mathit{free}((\forall x_4(\mathsf{R}(x_4, x_2) \lor (\mathsf{f}(x_3) = x_4)) \lor \exists x_3 \mathsf{S}(x_3, x_2)))$

$=$

Closed Formulas/Sentences

Note: Let φ be a formula and let α and β variable assignments with $\alpha(x) = \beta(x)$ for all free variables x of φ.
Then $\mathcal{I}, \alpha \models \varphi$ iff $\mathcal{I}, \beta \models \varphi$.

In particular, α is completely irrelevant if $\mathit{free}(\varphi) = \emptyset$.

Definition (Closed Formulas/Sentences)

A formula φ without free variables (i. e., $\mathit{free}(\varphi) = \emptyset$) is called closed formula or sentence.

If φ is a sentence, then we often write $\mathcal{I} \models \varphi$ instead of $\mathcal{I}, \alpha \models \varphi$, since the definition of α does not influence whether φ is true under $\mathcal{I}$ and α or not.

Formulas with at least one free variable are called open.

Closed formulas with no quantifiers are called ground formulas.

German: geschlossene Formel/Satz, offene Formel, Grundformel/variablenfreie Formel

Closed Formulas/Sentences: Examples

Question: Which of the following formulas are sentences?

- $(\mathsf{Block}(\mathsf{b}) \lor \neg \mathsf{Block}(\mathsf{b}))$
- $(\mathsf{Block}(x) \rightarrow (\mathsf{Block}(x) \lor \neg \mathsf{Block}(y)))$
- $(\mathsf{Block}(\mathsf{a}) \land \mathsf{Block}(\mathsf{b}))$
- $\forall x(\mathsf{Block}(x) \rightarrow \mathsf{Red}(x))$

D6.2 Reasoning in Predicate Logic

Terminology for Formulas

The terminology we introduced for propositional logic equally applies to predicate logic:

- Interpretation $\mathcal{I}$ and variable assignment α form a model of the formula φ if $\mathcal{I}, \alpha \models \varphi$.
- Formula φ is satisfiable if $\mathcal{I}, \alpha \models \varphi$ for at least one $\mathcal{I}, \alpha$.
- Formula φ is falsifiable if $\mathcal{I}, \alpha \not\models \varphi$. for at least one $\mathcal{I}, \alpha$
- Formula φ is valid if $\mathcal{I}, \alpha \models \varphi$ for all $\mathcal{I}, \alpha$.
- Formula φ is unsatisfiable if $\mathcal{I}, \alpha \not\models \varphi$ for all $\mathcal{I}, \alpha$.

German: Modell, erfüllbar, falsifizierbar, gültig, unerfüllbar

All concepts can be used for the special case of sentences. In this case we usually omit α. Examples:

- Interpretation $\mathcal{I}$ is a model of a sentence φ if $\mathcal{I} \models \varphi$.
- Sentence φ is unsatisfiable if $\mathcal{I} \not\models \varphi$ for all $\mathcal{I}$.

Sets of Formulas: Semantics

Definition (Satisfied/True Sets of Formulas)

Let $\mathcal{S}$ be a signature, Φ a set of formulas over $\mathcal{S}$,
$\mathcal{I}$ an interpretation for $\mathcal{S}$ and α a variable assignment for $\mathcal{S}$
and the universe of $\mathcal{I}$.

We say that $\mathcal{I}$ and α satisfy the formulas Φ
(also: Φ is true under $\mathcal{I}$ and α), written as: $\mathcal{I}, \alpha \models \Phi$,
if $\mathcal{I}, \alpha \models \varphi$ for all $\varphi \in \Phi$.

German: $\mathcal{I}$ und α erfüllen Φ, Φ ist wahr unter $\mathcal{I}$ und α

We may again write $\mathcal{I} \models \Phi$ if all formulas in Φ are sentences.

Logical Equivalence and Logical Consequences

We again we use the same concepts and notations as in propositional logic.

- A set of formulas Φ logically entails/implies formula ψ, written as $\Phi \models \psi$, if all models of Φ are models of ψ.
- For a single formula φ, we may write $\varphi \models \psi$ for $\{\varphi\} \models \psi$.
- Formulas φ and ψ are logically equivalent, written as $\varphi \equiv \psi$, if they have the same models.
  - Note that $\varphi \equiv \psi$ iff $\varphi \models \psi$ and $\psi \models \varphi$.

Important Theorems about Logical Consequences

Theorem (Deduction Theorem)

$\mathrm{KB} \cup \{\varphi\} \models \psi$ *iff* $\mathrm{KB} \models (\varphi \rightarrow \psi)$

German: Deduktionssatz

Theorem (Contraposition Theorem)

$\mathrm{KB} \cup \{\varphi\} \models \neg\psi$ *iff* $\mathrm{KB} \cup \{\psi\} \models \neg\varphi$

German: Kontrapositionssatz

Theorem (Contradiction Theorem)

$\mathrm{KB} \cup \{\varphi\}$ *is unsatisfiable iff* $\mathrm{KB} \models \neg\varphi$

German: Widerlegungssatz

These can be proved exactly the same way as in propositional logic.

Logical Equivalences

- All logical equivalences of propositional logic also hold in predicate logic (e. g., $(\varphi \lor \psi) \equiv (\psi \lor \varphi)$). (Why?)
- Additionally the following equivalences and implications hold:

$$
\begin{aligned}
(\forall x\varphi \land \forall x\psi) &\equiv \forall x(\varphi \land \psi) && \\
(\forall x\varphi \lor \forall x\psi) &\models \forall x(\varphi \lor \psi) && \text{but not the converse} \\
(\forall x\varphi \land \psi) &\equiv \forall x(\varphi \land \psi) && \text{if } x \notin \mathit{free}(\psi) \\
(\forall x\varphi \lor \psi) &\equiv \forall x(\varphi \lor \psi) && \text{if } x \notin \mathit{free}(\psi) \\
\neg\forall x\varphi &\equiv \exists x\neg\varphi && \\
\exists x(\varphi \lor \psi) &\equiv (\exists x\varphi \lor \exists x\psi) && \\
\exists x(\varphi \land \psi) &\models (\exists x\varphi \land \exists x\psi) && \text{but not the converse} \\
(\exists x\varphi \lor \psi) &\equiv \exists x(\varphi \lor \psi) && \text{if } x \notin \mathit{free}(\psi) \\
(\exists x\varphi \land \psi) &\equiv \exists x(\varphi \land \psi) && \text{if } x \notin \mathit{free}(\psi) \\
\neg\exists x\varphi &\equiv \forall x\neg\varphi &&
\end{aligned}
$$

Normal Forms (1)

Analogously to DNF and CNF for propositional logic there are several normal forms for predicate logic, such as

- negation normal form (NNF):
  negation symbols ($\neg$) are only allowed in front of atoms or identities
- prenex normal form:
  quantifiers must form the outermost part of the formula
- Skolem normal form:
  prenex normal form without existential quantifiers

German: Negationsnormalform, Pränexnormalform, Skolemnormalform

Normal Forms (2)

Efficient methods transform formula φ

- into an equivalent formula in negation normal form,
- into an equivalent formula in prenex normal form, or
- into an equisatisfiable formula in Skolem normal form.

German: erfüllbarkeitsäquivalent

Inference Rules and Calculi

There exist correct and complete proof systems (calculi) for predicate logic.

- An example is the natural deduction calculus.
- This is (essentially) Gödel's Completeness Theorem (1929).
- However, one can show that correct and complete algorithms that prove that a given formula does not follow from a given set of formulas cannot exist.
- How are these statements reconcilable?

D6.3 Summary and Outlook

Summary

- Predicate logic is more expressive than propositional logic and allows statements over objects and their properties.
- Objects are described by terms that are built from variable, constant and function symbols.
- Properties and relations are described by formulas that are built from predicates, quantifiers and the usual logical operators.
- Bound vs. free variables: to decide if $\mathcal{I}, \alpha \models \varphi$, only free variables in α matter
- Sentences (closed formulas): formulas without free variables

Summary

Once the basic definitions are in place, predicate logic can be developed in the same way as propositional logic:

- logical consequence
- deduction theorem etc.
- logical equivalences
- normal forms
- inference rules, proof systems, resolution

Other Logics (1)

- We considered first-order predicate logic.
- Second-order predicate logic allows quantifying over predicate symbols.
- There are intermediate steps, e. g., monadic second-order logic (all quantified predicates are unary) and description logics (foundation of the semantic web).

Second-Order Logic Example

Second-order logic example:

- "T is the transitive closure of R"
- conjunction of
  - $\forall x \forall y (R(x, y) \rightarrow T(x, y))$
    "T is a superset of R"
  - $\forall x \forall y \forall z ((T(x, y) \land T(y, z)) \rightarrow T(x, z))$
    "T is transitive"
  - $\forall Q((\forall x \forall y (R(x, y) \rightarrow Q(x, y)) \land$
    $\forall x \forall y \forall z ((Q(x, y) \land Q(y, z)) \rightarrow Q(x, z)))$
    $\rightarrow \forall x \forall y (T(x, y) \rightarrow Q(x, y))))$
    "All supersets Q of R that are transitive are supersets of T"
- impossible to express in first-order logic

Other Logics (2)

- Modal logics have new operators $\Box$ and $\Diamond$.
  - classical meaning: $\Box\varphi$ for "φ is necessary",
    $\Diamond\varphi$ for "φ is possible".
  - temporal logic: $\Box\varphi$ for "φ is always true in the future",
    $\Diamond\varphi$ for "φ is true at some point in the future"
  - epistemic logic: $\Box\varphi$ for "φ is known",
    $\Diamond\varphi$ for "φ is possible"
  - doxastic logic: $\Box\varphi$ for "φ is believed",
    $\Diamond\varphi$ for "φ is considered possible"
  - deontic logic: $\Box\varphi$ for "φ is obligatory",
    $\Diamond\varphi$ for "φ is permitted"
  - ...
- very important in computer-aided verification

Other Logics (3)

- In fuzzy logic, formulas are not true or false but have values between 0 and 1.
- Intuitionist logic is "constructive" and excludes indirect proof methods such as the principle of the excluded third.
- Non-monotonic logics have rules with exceptions (e.g., default logic, cumulative logic).
- ... and there is a lot more