Discrete Mathematics in Computer Science

D4. Inference

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Inference Rules and Calculi

Inference: Motivation

- up to now: proof of logical consequence with semantic arguments
- no general algorithm
- solution: produce formulas that are logical consequences of given formulas with syntactic inference rules
- advantage: mechanical method that can easily be implemented as an algorithm

Inference Rules

- Inference rules have the form

$$\frac{\varphi_1, \dots, \varphi_k}{\psi}.$$

- Meaning: "Every model of $\varphi_1, \dots, \varphi_k$ is a model of ψ."
- An axiom is an inference rule with $k = 0$.
- A set of inference rules is called a calculus or proof system.

German: Inferenzregel, Axiom, (der) Kalkül, Beweissystem

Some Inference Rules for Propositional Logic

Modus ponens $\dfrac{\varphi,\ (\varphi \to \psi)}{\psi}$

Modus tollens $\dfrac{\neg\psi,\ (\varphi \to \psi)}{\neg\varphi}$

$\wedge$-elimination $\dfrac{(\varphi \wedge \psi)}{\varphi}$ $\dfrac{(\varphi \wedge \psi)}{\psi}$

$\wedge$-introduction $\dfrac{\varphi,\ \psi}{(\varphi \wedge \psi)}$

$\vee$-introduction $\dfrac{\varphi}{(\varphi \vee \psi)}$

$\leftrightarrow$-elimination $\dfrac{(\varphi \leftrightarrow \psi)}{(\varphi \to \psi)}$ $\dfrac{(\varphi \leftrightarrow \psi)}{(\psi \to \varphi)}$

Derivation

Definition (Derivation)

A derivation or proof of a formula φ from a knowledge base KB is a sequence of formulas $\psi_1, \dots, \psi_k$ with

- $\psi_k = \varphi$ and
- for all $i \in \{1, \dots, k\}$:
  - $\psi_i \in \text{KB}$, or
  - ψ_i is the result of the application of an inference rule to elements from $\{\psi_1, \dots, \psi_{i-1}\}$.

German: Ableitung, Beweis

Derivation: Example

Example

Given: $\text{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$
Task: Find derivation of $(S \wedge R)$ from KB.

Derivation: Example

Example

Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)

Derivation: Example

Example

Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$
Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)

Derivation: Example

Example

Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$
Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)

Derivation: Example

Example

Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$
Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)

Derivation: Example

Example

Given: $\text{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)
5. R (1, 4, Modus ponens)

Derivation: Example

Example

Given: $\text{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)
5. R (1, 4, Modus ponens)
6. $(Q \wedge R)$ (3, 5, $\wedge$-introduction)

Derivation: Example

Example

Given: $\text{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)
5. R (1, 4, Modus ponens)
6. $(Q \wedge R)$ (3, 5, $\wedge$-introduction)
7. $((Q \wedge R) \rightarrow S)$ (KB)

Derivation: Example

Example

Given: $\mathrm{KB} = \{P, (P \rightarrow Q), (P \rightarrow R), ((Q \wedge R) \rightarrow S)\}$

Task: Find derivation of $(S \wedge R)$ from KB.

1. P (KB)
2. $(P \rightarrow Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \rightarrow R)$ (KB)
5. R (1, 4, Modus ponens)
6. $(Q \wedge R)$ (3, 5, $\wedge$-introduction)
7. $((Q \wedge R) \rightarrow S)$ (KB)
8. S (6, 7, Modus ponens)

Derivation: Example

Example

Given: $\mathrm{KB} = \{P, (P \to Q), (P \to R), ((Q \land R) \to S)\}$
Task: Find derivation of $(S \land R)$ from KB.

1. P (KB)
2. $(P \to Q)$ (KB)
3. Q (1, 2, Modus ponens)
4. $(P \to R)$ (KB)
5. R (1, 4, Modus ponens)
6. $(Q \land R)$ (3, 5, $\land$-introduction)
7. $((Q \land R) \to S)$ (KB)
8. S (6, 7, Modus ponens)
9. $(S \land R)$ (8, 5, $\land$-introduction)

Correctness and Completeness

Definition (Correctness and Completeness of a Calculus)

We write $\mathrm{KB} \vdash_C \varphi$ if there is a derivation of φ from KB in calculus C.
(If calculus C is clear from context, also only $\mathrm{KB} \vdash \varphi$.)

A calculus C is correct if for all KB and φ
$\mathrm{KB} \vdash_C \varphi$ implies $\mathrm{KB} \models \varphi$.

A calculus C is complete if for all KB and φ
$\mathrm{KB} \models \varphi$ implies $\mathrm{KB} \vdash_C \varphi$.

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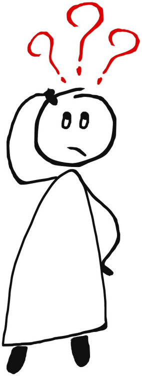
Consider calculus C, consisting of the derivation rules seen earlier.
Question: Is C correct?
Question: Is C complete?

German: korrekt, vollständig

Questions



Questions?

Summary

Summary (Consequence and Inference)

- knowledge base: set of formulas describing given information; satisfiable, valid etc. used like for individual formulas
- logical consequence $\text{KB} \models \varphi$ means that φ is true whenever (= in all models where) KB is true
- A logical consequence $\text{KB} \models \varphi$ allows to conclude that KB implies φ based on the semantics.
- A correct calculus supports such conclusions on the basis of purely syntactical derivations $\text{KB} \vdash \varphi$.

Further Topics

There are many aspects of propositional logic
that we do not cover in this course.

- resolution: a commonly used proof system for formulas in CNF
- other proof systems, for example tableaux proofs
- algorithms for model construction, such as the Davis-Putnam-Logemann-Loveland (DPLL) algorithm.

$\rightsquigarrow$ Foundations of AI course