

Discrete Mathematics in Computer Science

D3. Normal Forms and Logical Consequence

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Simplified Notation

Parentheses

Associativity:

$$((\varphi \wedge \psi) \wedge \chi) \equiv (\varphi \wedge (\psi \wedge \chi))$$

$$((\varphi \vee \psi) \vee \chi) \equiv (\varphi \vee (\psi \vee \chi))$$

- Placement of parentheses for a conjunction of conjunctions does not influence whether an interpretation is a model.
- ditto for disjunctions of disjunctions
- ~> can omit parentheses and treat this as if parentheses placed arbitrarily
- **Example:** $(A_1 \wedge A_2 \wedge A_3 \wedge A_4)$ instead of $((A_1 \wedge (A_2 \wedge A_3)) \wedge A_4)$
- **Example:** $(\neg A \vee (B \wedge C) \vee D)$ instead of $((\neg A \vee (B \wedge C)) \vee D)$

Parentheses

Does this mean we can always omit all parentheses and assume an arbitrary placement? \rightsquigarrow **No!**

$$((\varphi \wedge \psi) \vee \chi) \not\equiv (\varphi \wedge (\psi \vee \chi))$$

What should $\varphi \wedge \psi \vee \chi$ mean?

Placement of Parentheses by Convention

Often parentheses can be dropped in specific cases and an **implicit** placement is assumed:

- \neg binds more strongly than \wedge
- \wedge binds more strongly than \vee
- \vee binds more strongly than \rightarrow or \leftrightarrow

\rightsquigarrow cf. PEMDAS/“Punkt vor Strich”

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- often harder to read
- error-prone

\rightsquigarrow not used in this course

Short Notations for Conjunctions and Disjunctions

Short notation for addition:

$$\sum_{i=1}^n x_i = x_1 + x_2 + \cdots + x_n$$

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$$\sum_{i=1}^n x_i = x_1 + x_2 + \cdots + x_n$$
$$\sum_{x \in \{x_1, \dots, x_n\}} x = x_1 + x_2 + \cdots + x_n$$

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Analogously (possible because of commutativity of \wedge and \vee):

$$\bigwedge_{i=1}^n \varphi_i = (\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)$$
$$\bigvee_{i=1}^n \varphi_i = (\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n)$$
$$\bigwedge_{\varphi \in X} \varphi = (\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)$$
$$\bigvee_{\varphi \in X} \varphi = (\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n)$$

for $X = \{\varphi_1, \dots, \varphi_n\}$

Short Notation: Corner Cases

Is $\mathcal{I} \models \psi$ true for

$$\psi = \bigwedge_{\varphi \in X} \varphi \text{ and } \psi = \bigvee_{\varphi \in X} \varphi$$

if $X = \emptyset$ or $X = \{\chi\}$?

Short Notation: Corner Cases

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convention:

- $\bigwedge_{\varphi \in \emptyset} \varphi$ is a tautology.
- $\bigvee_{\varphi \in \emptyset} \varphi$ is unsatisfiable.
- $\bigwedge_{\varphi \in \{\chi\}} \varphi = \bigvee_{\varphi \in \{\chi\}} \varphi = \chi$

\rightsquigarrow Why?

Exercise

Express $\bigwedge_{i=1}^2 \bigvee_{j=1}^3 \varphi_{ij}$ without \bigwedge and \bigvee .

Normal Forms

Why Normal Forms?

- A **normal form** is a representation with **certain syntactic restrictions**.
- condition for reasonable normal form: **every formula** must have a logically **equivalent formula in normal form**
- **advantages:**
 - can restrict proofs to formulas in normal form
 - can define algorithms to work only for formulas in normal form

German: Normalform

Negation Normal Form

Definition (Negation Normal Form)

A formula is in **negation normal form (NNF)** if it does not contain the abbreviations \rightarrow and \leftrightarrow and if it contains no negation symbols except possibly directly in front of atomic propositions.

German: Negationsnormalform

Example

$((\neg P \vee (R \wedge Q)) \wedge (P \vee \neg S))$ is in NNF.

$(P \wedge \neg(Q \vee R))$ is not in NNF.

Construction of NNF

Algorithm to Construct NNF

- ① Replace abbreviation \leftrightarrow by its definition ((\leftrightarrow) -elimination).
 \rightsquigarrow formula structure: only \neg , \vee , \wedge , \rightarrow
- ② Replace abbreviation \rightarrow by its definition ((\rightarrow) -elimination).
 \rightsquigarrow formula structure: only \neg , \vee , \wedge
- ③ Repeatedly apply **double negation** and **De Morgan** rules until no rules match any more (“move negations inside”):
 - Replace $\neg\neg\varphi$ by φ .
 - Replace $\neg(\varphi \wedge \psi)$ by $(\neg\varphi \vee \neg\psi)$.
 - Replace $\neg(\varphi \vee \psi)$ by $(\neg\varphi \wedge \neg\psi)$. \rightsquigarrow formula structure: only atoms, negated atoms, \vee , \wedge

Constructing NNF: Example

Construction of Negation Normal Form

Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

Constructing NNF: Example

Construction of Negation Normal Form

Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

$$\varphi \equiv (\neg((P \wedge \neg Q) \vee R) \vee P \vee \neg(S \vee T)) \quad [\text{Step 2}]$$

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Construction of Negation Normal Form

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$$\equiv ((\neg(P \wedge \neg Q) \wedge \neg R) \vee P \vee \neg(S \vee T)) \quad [\text{Step 3}]$$

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Literals, Clauses and Monomials

- A **literal** is an atomic proposition or the negation of an atomic proposition (e. g., A and $\neg A$).
- A **clause** is a disjunction of literals (e. g., $(Q \vee \neg P \vee \neg S \vee R)$).
- A **monomial** is a conjunction of literals (e. g., $(Q \wedge \neg P \wedge \neg S \wedge R)$).

The terms **clause** and **monomial** are also used for the corner case with **only one literal**.

German: Literal, Klausel, Monom

Terminology: Examples

Examples

- $(\neg Q \wedge R)$
- $(P \vee \neg Q)$
- $((P \vee \neg Q) \wedge P)$
- $\neg P$
- $(P \rightarrow Q)$

- $(P \vee P)$
- $\neg\neg P$

Terminology: Examples

Examples

- $(\neg Q \wedge R)$ is a monomial
- $(P \vee \neg Q)$
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Terminology: Examples

Examples

- $(\neg Q \wedge R)$ is a monomial
- $(P \vee \neg Q)$ is a clause
- $((P \vee \neg Q) \wedge P)$ is neither literal nor clause nor monomial
- $\neg P$
- $(P \rightarrow Q)$
- $(P \vee P)$
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Terminology: Examples

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- $(P \vee P)$ is a clause, but not a literal or monomial
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- $\neg\neg P$ is neither literal nor clause nor monomial

Conjunctive Normal Form

Definition (Conjunctive Normal Form)

A formula is in **conjunctive normal form (CNF)** if it is a conjunction of clauses, i. e., if it has the form

$$\bigwedge_{i=1}^n \bigvee_{j=1}^{m_i} L_{ij}$$

with $n, m_i > 0$ (for $1 \leq i \leq n$), where the L_{ij} are literals.

German: konjunktive Normalform (KNF)

Example

$((\neg P \vee Q) \wedge R \wedge (P \vee \neg S))$ is in CNF.

Disjunctive Normal Form

Definition (Disjunctive Normal Form)

A formula is in **disjunctive normal form (DNF)** if it is a disjunction of monomials, i. e., if it has the form

$$\bigvee_{i=1}^n \bigwedge_{j=1}^{m_i} L_{ij}$$

with $n, m_i > 0$ (for $1 \leq i \leq n$), where the L_{ij} are literals.

German: disjunktive Normalform (DNF)

Example

$((\neg P \wedge Q) \vee R \vee (P \wedge \neg S))$ is in DNF.

NNF, CNF and DNF: Examples

Which of the following formulas are in NNF?

Which are in CNF? Which are in DNF?

- $((P \vee \neg Q) \wedge P)$
- $((R \vee Q) \wedge P \wedge (R \vee S))$
- $(P \vee (\neg Q \wedge R))$
- $(P \vee \neg\neg Q)$
- $(P \rightarrow \neg Q)$

- $((P \vee \neg Q) \rightarrow P)$
- P

NNF, CNF and DNF: Examples

Which of the following formulas are in NNF?

Which are in CNF? Which are in DNF?

- $((P \vee \neg Q) \wedge P)$ is in NNF and CNF
- $((R \vee Q) \wedge P \wedge (R \vee S))$
- $(P \vee (\neg Q \wedge R))$
- $(P \vee \neg\neg Q)$
- $(P \rightarrow \neg Q)$

- $((P \vee \neg Q) \rightarrow P)$
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- $(P \vee (\neg Q \wedge R))$ is in NNF and DNF
- $(P \vee \neg\neg Q)$ is in none of the normal forms
- $(P \rightarrow \neg Q)$
- $((P \vee \neg Q) \rightarrow P)$
- P

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- $(P \vee (\neg Q \wedge R))$ is in NNF and DNF
- $(P \vee \neg\neg Q)$ is in none of the normal forms
- $(P \rightarrow \neg Q)$ is in none of the normal forms,
but is in all three after expanding \rightarrow
- $((P \vee \neg Q) \rightarrow P)$
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NNF, CNF and DNF: Examples

Which of the following formulas are in NNF?

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- $((P \vee \neg Q) \wedge P)$ is in NNF and CNF
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- $(P \vee (\neg Q \wedge R))$ is in NNF and DNF
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but is in all three after expanding \rightarrow
- $((P \vee \neg Q) \rightarrow P)$ is in none of the normal forms
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NNF, CNF and DNF: Examples

Which of the following formulas are in NNF?

Which are in CNF? Which are in DNF?

- $((P \vee \neg Q) \wedge P)$ is in NNF and CNF
- $((R \vee Q) \wedge P \wedge (R \vee S))$ is in NNF and CNF
- $(P \vee (\neg Q \wedge R))$ is in NNF and DNF
- $(P \vee \neg\neg Q)$ is in none of the normal forms
- $(P \rightarrow \neg Q)$ is in none of the normal forms,
but is in all three after expanding \rightarrow
- $((P \vee \neg Q) \rightarrow P)$ is in none of the normal forms
- P is in NNF, CNF and DNF

Construction of CNF (and DNF)

Algorithm to Construct CNF

First, convert to NNF (steps 1–3).

\rightsquigarrow formula structure: only literals, \vee , \wedge

- ④ Repeatedly apply **distributivity** or **commutativity + distributivity** to distribute \vee over \wedge :

- Replace $(\varphi \vee (\psi \wedge \chi))$ by $((\varphi \vee \psi) \wedge (\varphi \vee \chi))$.
- Replace $((\psi \wedge \chi) \vee \varphi)$ by $((\psi \vee \varphi) \wedge (\chi \vee \varphi))$.

\rightsquigarrow formula structure: CNF

- ⑤ **optionally**: Simplify the formula at the end or at intermediate steps (e. g., with idempotence).

Note: For DNF, swap the roles of \wedge and \vee in Step 4.

Constructing CNF: Example

Construction of Conjunctive Normal Form

Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

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$$\varphi \equiv (((\neg P \vee Q) \wedge \neg R) \vee P \vee (\neg S \wedge \neg T)) \text{ [to NNF]}$$

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$$\varphi \equiv (((\neg P \vee Q) \wedge \neg R) \vee P \vee (\neg S \wedge \neg T)) \text{ [to NNF]}$$

$$\equiv ((\neg P \vee Q \vee P \vee (\neg S \wedge \neg T)) \wedge$$

$$(\neg R \vee P \vee (\neg S \wedge \neg T))) \quad \text{[Step 4]}$$

Constructing CNF: Example

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$$\equiv ((\neg P \vee Q \vee P \vee (\neg S \wedge \neg T)) \wedge$$

$$(\neg R \vee P \vee (\neg S \wedge \neg T))) \quad \text{[Step 4]}$$

$$\equiv (\neg R \vee P \vee (\neg S \wedge \neg T)) \quad \text{[Step 5]}$$

Constructing CNF: Example

Construction of Conjunctive Normal Form

Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

$$\varphi \equiv (((\neg P \vee Q) \wedge \neg R) \vee P \vee (\neg S \wedge \neg T)) \text{ [to NNF]}$$

$$\equiv ((\neg P \vee Q \vee P \vee (\neg S \wedge \neg T)) \wedge$$

$$(\neg R \vee P \vee (\neg S \wedge \neg T))) \quad \text{[Step 4]}$$

$$\equiv (\neg R \vee P \vee (\neg S \wedge \neg T)) \quad \text{[Step 5]}$$

$$\equiv ((\neg R \vee P \vee \neg S) \wedge (\neg R \vee P \vee \neg T)) \quad \text{[Step 4]}$$

Construct DNF: Example

Construction of Disjunctive Normal Form

Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

$$\varphi \equiv (((\neg P \vee Q) \wedge \neg R) \vee P \vee (\neg S \wedge \neg T)) \quad [\text{to NNF}]$$

$$\equiv ((\neg P \wedge \neg R) \vee (Q \wedge \neg R) \vee P \vee (\neg S \wedge \neg T)) \quad [\text{Step 4}]$$

Existence of an Equivalent Formula in Normal Form

Theorem

For every formula φ there is a logically equivalent formula in NNF, a logically equivalent formula in CNF and a logically equivalent formula in DNF.

- “There is a” always means “there is at least one”. Otherwise we would write “there is exactly one”.
- Intuition: algorithms to construct normal forms work with any given formula and only use equivalence rewriting.
- actual proof would use induction over structure of formula

Size of Normal Forms

- In the worst case, a logically equivalent formula in CNF or DNF can be exponentially larger than the original formula.
- **Example:** for $(x_1 \vee y_1) \wedge \cdots \wedge (x_n \vee y_n)$ there is no smaller logically equivalent formula in DNF than:

$$\bigvee_{S \in \mathcal{P}(\{1, \dots, n\})} \left(\bigwedge_{i \in S} x_i \wedge \bigwedge_{i \in \{1, \dots, n\} \setminus S} y_i \right)$$

- As a consequence, the construction of the CNF/DNF formula can take exponential time.
- For NNF, we can generate an equivalent formula in linear time if the original formula does not use \leftrightarrow .

More Theorems

Theorem

A formula in CNF is a tautology iff every clause is a tautology.

Theorem

A formula in DNF is satisfiable iff at least one of its monomials is satisfiable.

\leadsto both proved easily with semantics of propositional logic

Knowledge Bases

Knowledge Bases: Example



If not DrinkBeer, then EatFish.
If EatFish and DrinkBeer,
then not EatIceCream.
If EatIceCream or not DrinkBeer,
then not EatFish.

$$\text{KB} = \{(\neg \text{DrinkBeer} \rightarrow \text{EatFish}),$$
$$((\text{EatFish} \wedge \text{DrinkBeer}) \rightarrow \neg \text{EatIceCream}),$$
$$((\text{EatIceCream} \vee \neg \text{DrinkBeer}) \rightarrow \neg \text{EatFish})\}$$

Models for Sets of Formulas

Definition (Model for Knowledge Base)

Let KB be a **knowledge base** over A ,
i. e., a set of propositional formulas over A .

A truth assignment \mathcal{I} for A is a **model for KB** (written: $\mathcal{I} \models \text{KB}$)
if \mathcal{I} is a **model for every formula** $\varphi \in \text{KB}$.

German: Wissensbasis, Modell

Properties of Sets of Formulas

A knowledge base KB is

- **satisfiable** if KB has at least one model
- **unsatisfiable** if KB is not satisfiable
- **valid** (or a **tautology**) if every interpretation is a model for KB
- **falsifiable** if KB is no tautology

German: erfüllbar, unerfüllbar, gültig, gültig/eine Tautologie, falsifizierbar

Example I

Which of the properties does $KB = \{(A \wedge \neg B), \neg(B \vee A)\}$ have?

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Which of the properties does $KB = \{(A \wedge \neg B), \neg(B \vee A)\}$ have?

KB is **unsatisfiable**:

For every model \mathcal{I} with $\mathcal{I} \models (A \wedge \neg B)$ we have $\mathcal{I}(A) = 1$.

This means $\mathcal{I} \models (B \vee A)$ and thus $\mathcal{I} \not\models \neg(B \vee A)$.

This directly implies that KB is **falsifiable**, **not satisfiable** and **no tautology**.

Example II

Which of the properties does

$KB = \{(\neg \text{DrinkBeer} \rightarrow \text{EatFish}),$
 $((\text{EatFish} \wedge \text{DrinkBeer}) \rightarrow \neg \text{EatIceCream}),$
 $((\text{EatIceCream} \vee \neg \text{DrinkBeer}) \rightarrow \neg \text{EatFish})\}$ have?

Example II

Which of the properties does

$KB = \{(\neg \text{DrinkBeer} \rightarrow \text{EatFish}),$
 $((\text{EatFish} \wedge \text{DrinkBeer}) \rightarrow \neg \text{EatIceCream}),$
 $((\text{EatIceCream} \vee \neg \text{DrinkBeer}) \rightarrow \neg \text{EatFish})\}$ have?

- **satisfiable**, e. g. with

$\mathcal{I} = \{\text{EatFish} \mapsto 1, \text{DrinkBeer} \mapsto 1, \text{EatIceCream} \mapsto 0\}$

- thus **not unsatisfiable**

- **falsifiable**, e. g. with

$\mathcal{I} = \{\text{EatFish} \mapsto 0, \text{DrinkBeer} \mapsto 0, \text{EatIceCream} \mapsto 1\}$

- thus **not valid**

Logical Consequences

Logical Consequences: Motivation

What's the secret of your long life?



I am on a strict diet: If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

Claim: the woman drinks beer to every meal.

How can we prove this?

Logical Consequences

Definition (Logical Consequence)

Let KB be a set of formulas and φ a formula.

We say that KB **logically implies** φ (written as $\text{KB} \models \varphi$) if **all models** of KB are also models of φ .

also: KB **logically entails** φ , φ **logically follows** from KB, φ is a **logical consequence** of KB

German: KB impliziert φ logisch, φ folgt logisch aus KB, φ ist logische Konsequenz von KB

Attention: the symbol \models is “overloaded”: $\text{KB} \models \varphi$ vs. $\mathcal{I} \models \varphi$.

What if KB is unsatisfiable or the empty set?

Logical Consequences: Example

Let $\varphi = \text{DrinkBeer}$ and

$$\begin{aligned} \text{KB} = \{ & (\neg \text{DrinkBeer} \rightarrow \text{EatFish}), \\ & ((\text{EatFish} \wedge \text{DrinkBeer}) \rightarrow \neg \text{EatIceCream}), \\ & ((\text{EatIceCream} \vee \neg \text{DrinkBeer}) \rightarrow \neg \text{EatFish}) \}. \end{aligned}$$

Show: $\text{KB} \models \varphi$

Logical Consequences: Example

Let $\varphi = \text{DrinkBeer}$ and

$$\begin{aligned} \text{KB} = \{ & (\neg \text{DrinkBeer} \rightarrow \text{EatFish}), \\ & ((\text{EatFish} \wedge \text{DrinkBeer}) \rightarrow \neg \text{EatIceCream}), \\ & ((\text{EatIceCream} \vee \neg \text{DrinkBeer}) \rightarrow \neg \text{EatFish}) \}. \end{aligned}$$

Show: $\text{KB} \models \varphi$

Proof sketch.

Proof by contradiction: assume $\mathcal{I} \models \text{KB}$, but $\mathcal{I} \not\models \text{DrinkBeer}$.
Then it follows that $\mathcal{I} \models \neg \text{DrinkBeer}$.

Because \mathcal{I} is a model of KB, we also have
 $\mathcal{I} \models (\neg \text{DrinkBeer} \rightarrow \text{EatFish})$ and thus $\mathcal{I} \models \text{EatFish}$. (Why?)

With an analogous argumentation starting from
 $\mathcal{I} \models ((\text{EatIceCream} \vee \neg \text{DrinkBeer}) \rightarrow \neg \text{EatFish})$
we get $\mathcal{I} \models \neg \text{EatFish}$ and thus $\mathcal{I} \not\models \text{EatFish}$. \rightsquigarrow **Contradiction!**

Important Theorems about Logical Consequences

Theorem (Deduction Theorem)

$KB \cup \{\varphi\} \models \psi$ *iff* $KB \models (\varphi \rightarrow \psi)$

German: Deduktionssatz

Theorem (Contraposition Theorem)

$KB \cup \{\varphi\} \models \neg\psi$ *iff* $KB \cup \{\psi\} \models \neg\varphi$

German: Kontrapositionssatz

Theorem (Contradiction Theorem)

$KB \cup \{\varphi\}$ *is unsatisfiable* *iff* $KB \models \neg\varphi$

German: Widerlegungssatz

(without proof)