# Discrete Mathematics in Computer Science

## C3. Acyclicity

Malte Helmert, Gabriele Röger

University of Basel

November 10/12, 2025

# Acyclic (Di-) Graphs

# Acyclic

Similarly to connectedness, the presence or absence of cycles is an important practical property for (di-) graphs.

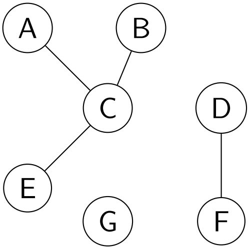
**Definition (acyclic, forest, DAG)**

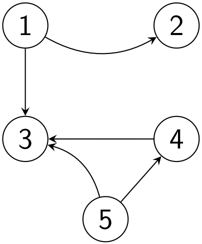
A graph or digraph $G$ is called acyclic if there exists no cycle in $G$.

An acyclic graph is also called a forest.
An acyclic digraph is also called a DAG (directed acyclic graph).

German: azyklisch/kreisfrei, Wald, DAG

# Acyclic (Di-) Graphs – Example



# Trees

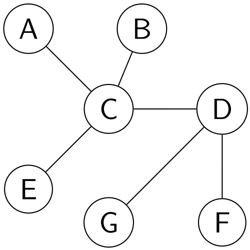
**Definition (tree)**

A connected forest is called a tree.

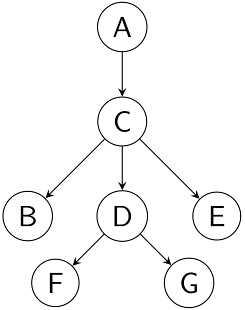
German: Baum

- Tree is also a word for a recursive data structure, which consists of either a leaf or a parent node with one or more children, which are themselves trees.
- This other kind of tree is also called a rooted tree to distinguish it from a tree as a graph.
- The two meanings of "tree" are distinct but closely related.

# Tree Graphs vs. Rooted Trees – Example (1)

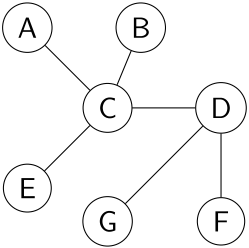



tree graph

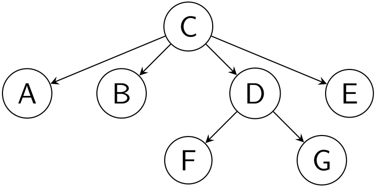



rooted tree with root $A$

# Tree Graphs vs. Rooted Trees – Example (2)

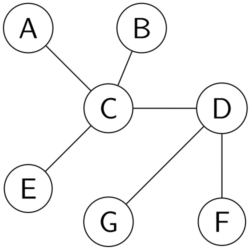



tree graph

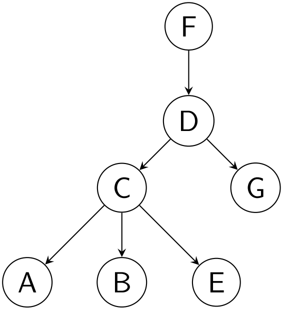



rooted tree with root $C$

# Tree Graphs vs. Rooted Trees – Example (3)




tree graph




rooted tree with root $F$

# From Tree Graphs to Rooted Trees

General procedure for converting tree graphs into rooted trees:

- Select any vertex $v$. Make $v$ the root of the tree.
- Initially, $v$ is the only pending vertex,
  and there are no processed vertices.
- As long as there are pending vertices:
  - Select any pending vertex $u$.
  - Make all neighbours $v$ of $u$ that are not yet processed children of $u$ and mark them as pending.
  - Change $u$ from pending to processed.

We do not prove that this procedure always works. A proof of correctness can be given based on the results we show next.

# Unique Paths in Trees

# Unique Paths in Trees

**Theorem**

*Let $G = (V, E)$ be a graph.*
*Then $G$ is a tree iff there exists exactly one path from any vertex $u \in V$ to any vertex $v \in V$.*

# Unique Paths In Trees – Proof (1)

**Proof.**

($\Rightarrow$): $G$ is a tree. Let $u, v \in V$.
We must show that there exists exactly one path from $u$ to $v$.

We know that at least one path exists because $G$ is connected.

It remains to show that there cannot be two paths from $u$ to $v$.
If $u = v$, there is only one path (the empty one).
(Any longer path would have to repeat a vertex.)

We assume that there exist two different paths from $u$ to $v$ ($u \neq v$) and derive a contradiction. . . .

# Unique Paths In Trees – Proof (2)

**Proof (continued).**

Let $\pi = \langle v_0, v_1, \ldots, v_n \rangle$ and $\pi' = \langle v'_0, v'_1, \ldots, v'_m \rangle$ be the two paths (with $v_0 = v'_0 = u$ and $v_n = v'_m = v$).
Let $i$ be the smallest index with $v_i \neq v'_i$, which must exist because the two paths are different, and neither can be a prefix of the other (else $v$ would be repeated in the longer path).
We have $i \geq 1$ because $v_0 = v'_0$.
Let $j \geq i$ be the smallest index such that $v_j = v'_k$ for some $k \geq i$.
Such an index must exist because $v_n = v'_m$.
Then $\langle v_{i-1}, \ldots, v_{j-1}, v'_k, \ldots, v'_{i-1} \rangle$ is a cycle,
which contradicts the requirement that $G$ is a tree. ...

# Unique Paths In Trees – Proof (3)

**Proof (continued).**

($\Leftarrow$): For all $u, v \in V$, there exists exactly one path from $u$ to $v$.
We must show that $G$ is a tree, i.e., is connected and acyclic.

Because there exist paths from all $u$ to all $v$, $G$ is connected.

Proof by contradiction: assume that there exists a cycle in $G$,
$\pi = \langle u, v_1, \ldots, v_n, u \rangle$ with $n \geq 2$.
(Note that all cycles have length at least 3.)
From the definition of cycles, we have $v_1 \neq v_n$.

Then $\langle u, v_1 \rangle$ and $\langle u, v_n, \ldots, v_1 \rangle$ are two different paths from $u$ to $v_1$, contradicting that there exists exactly one path from every vertex to every vertex. Hence $G$ must be acyclic. □

# Leaves and Edge Counts in Trees and Forests

# Leaves in Trees

**Definition**

Let $G = (V, E)$ be a tree.
A leaf of $G$ is a vertex $v \in V$ with $\deg(v) \leq 1$.

Note: The case $\deg(v) = 0$ only occurs in single-vertex trees ($|V| = 1$). In trees with at least two vertices, vertices with degree 0 cannot exist because this would make the graph unconnected.

**Theorem**

*Let $G = (V, E)$ be a tree with $|V| \geq 2$.*
*Then $G$ has at least two leaves.*

# Leaves in Trees – Proof

**Proof.**

Let $\pi = \langle v_0, \ldots, v_n \rangle$ be path in $G$ with maximal length among all paths in $G$.
Because $|V| \geq 2$, we have $n \geq 1$ (else $G$ would not be connected).

We show that vertex $v_n$ has degree 1: $v_{n-1}$ is a neighbour in $G$.
Assume that it were not the only neighbour of $v_n$ in $G$, so $u$ is another neighbour of $v_n$. Then:

- If $u$ is not on the path, then $\langle v_0, \ldots, v_n, u \rangle$ is a longer path: contradiction.
- If $u$ is on the path, then $u = v_i$ for some $i \neq n$ and $i \neq n - 1$. Then $\langle v_i, \ldots, v_n, v_i \rangle$ is a cycle: contradiction.

By reversing $\pi$ we can show $\deg(v_0) = 1$ in the same way. $\square$

# Edges in Trees

**Theorem**

*Let $G = (V, E)$ be a tree with $V \neq \emptyset$.*
*Then $|E| = |V| - 1$.*

# Edges in Trees – Proof (1)

**Proof.**

Proof by induction over $n = |V|$.

# Edges in Trees – Proof (1)

**Proof.**

Proof by induction over $n = |V|$.

Induction base ($n = 1$):
Then $G$ has 1 vertex and 0 edges.
We get $|E| = 0 = 1 - 1 = |V| - 1$.

# Edges in Trees – Proof (1)

**Proof.**

Proof by induction over $n = |V|$.

Induction base ($n = 1$):
Then $G$ has 1 vertex and 0 edges.
We get $|E| = 0 = 1 - 1 = |V| - 1$.

Induction step ($n \to n + 1$):
Let $G = (V, E)$ be a tree with $n + 1$ vertices ($n \geq 1$).
From the previous result, $G$ has a leaf $u$.
Let $v$ be the only neighbour of $u$.
Let $e = \{u, v\}$ be the connecting edge. ...

# Edges in Trees – Proof (2)

**Proof (continued).**

Consider the graph $G' = (V', E')$
with $V' = V \setminus \{u\}$ and $E' = E \setminus \{e\}$.

- $G'$ is acyclic: every cycle in $G'$ would also be present in $G$ (contradiction).
- $G'$ is connected: for all vertices $w \neq u$ and $w' \neq u$, $G$ has a path $\pi$ from $w$ to $w'$ because $G$ is connected. Path $\pi$ cannot include $u$ because $u$ has only one neighbour, so traversing $u$ requires repeating $v$. Hence $\pi$ is also a path in $G'$.

Hence $G'$ is a tree with $n$ vertices, and we can apply the induction hypothesis, which gives $|E'| = |V'| - 1$.
It follows that
$|E| = |E'| + 1 = (|V'| - 1) + 1 = (|V'| + 1) - 1 = |V| - 1.$ □

# Edges in Forests

**Theorem**

*Let $G = (V, E)$ be a forest.*
*Let $C$ be the set of connected components of $G$.*
*Then $|E| = |V| - |C|$.*

This result generalizes the previous one.

# Edges in Forests – Proof

**Proof.**

Let $C = \{C_1, \ldots, C_k\}$.

For $1 \leq i \leq k$, let $G_i = (C_i, E_i)$ be $G$ restricted to $C_i$, i.e.,
the graph whose vertices are $C_i$
and whose edges are the edges $e \in E$ with $e \subseteq C_i$.

We have $|V| = \sum_{i=1}^{k} |C_i|$ because the connected components form a partition of $V$.

We have $|E| = \sum_{i=1}^{k} |E_i|$ because every edge belongs to exactly one connected component. (Note that there cannot be edges between different connected components.)

Every graph $G_i$ is a tree with at least one vertex:
it is connected because its vertices form a connected component,
and it is acyclic because $G$ is acyclic. This implies $|E_i| = |C_i| - 1$.

Putting this together, we get
$|E| = \sum_{i=1}^{k} |E_i| = \sum_{i=1}^{k}(|C_i| - 1) = \sum_{i=1}^{k} |C_i| - k = |V| - |C|$. $\square$

# Characterizations of Trees

# Characterizations of Trees

**Theorem**

*Let $G = (V, E)$ be a graph with $V \neq \emptyset$.*
*The following statements are equivalent:*

1. *$G$ is a tree.*
2. *$G$ is acyclic and connected.*
3. *$G$ is acyclic and $|E| = |V| - 1$.*
4. *$G$ is connected and $|E| = |V| - 1$.*
5. *For all $u, v \in V$ there exists exactly one path from $u$ to $v$.*

# Characterizations of Trees – Proof (1)

Reminder:

(1) $G$ is a tree.
(2) $G$ is acyclic and connected.
(3) $G$ is acyclic and $|E| = |V| - 1$.
(4) $G$ is connected and $|E| = |V| - 1$.
(5) For all $u, v \in V$ there exists exactly one path from $u$ to $v$.

**Proof.**

We know already:

- (1) and (2) are equivalent by definition of trees.
- We have shown that (1) and (5) are equivalent.
- We have shown that (1) implies (3) and (4).

We complete the proof by showing (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (2). ...

# Characterizations of Trees – Proof (2)

Reminder:

(2) $G$ is acyclic and connected.

(3) $G$ is acyclic and $|E| = |V| - 1$.

## Proof (continued).

(3) $\Rightarrow$ (2):

Because $G$ is acyclic, it is a forest.

From the previous result, we have $|E| = |V| - |C|$,

where $C$ are the connected components of $G$.

# Characterizations of Trees – Proof (2)

Reminder:

(2) $G$ is acyclic and connected.

(3) $G$ is acyclic and $|E| = |V| - 1$.

**Proof (continued).**

(3) $\Rightarrow$ (2):

Because $G$ is acyclic, it is a forest.
From the previous result, we have $|E| = |V| - |C|$,
where $C$ are the connected components of $G$.

But we also know $|E| = |V| - 1$. This implies $|C| = 1$.
Hence $G$ is connected and therefore a tree. . . .

# Characterizations of Trees – Proof (3)

Reminder:
(2) $G$ is acyclic and connected.
(4) $G$ is connected and $|E| = |V| - 1$.

**Proof (continued).**

(4) $\Rightarrow$ (2):

In graphs that are not acyclic, we can remove an edge without changing the connected components: if $\langle v_0, \dots, v_n, v_0 \rangle$ ($n \geq 2$) is a cycle, remove the edge $\{v_0, v_1\}$ from the graph.
Every walk using this edge can substitute $\langle v_1, \dots, v_n, v_0 \rangle$ (or the reverse path) for it.

# Characterizations of Trees – Proof (3)

Reminder:

(2) $G$ is acyclic and connected.

(4) $G$ is connected and $|E| = |V| - 1$.

**Proof (continued).**

(4) $\Rightarrow$ (2):

In graphs that are not acyclic, we can remove an edge without changing the connected components: if $\langle v_0, \dots, v_n, v_0 \rangle$ $(n \geq 2)$ is a cycle, remove the edge $\{v_0, v_1\}$ from the graph.
Every walk using this edge can substitute $\langle v_1, \dots, v_n, v_0 \rangle$ (or the reverse path) for it.

Iteratively remove edges from $G$ in this way while preserving connectedness until this is no longer possible. The resulting graph $(V, E')$ is acyclic and connected and therefore a tree.

# Characterizations of Trees – Proof (3)

Reminder:

(2) $G$ is acyclic and connected.

(4) $G$ is connected and $|E| = |V| - 1$.

**Proof (continued).**

(4) $\Rightarrow$ (2):

In graphs that are not acyclic, we can remove an edge without changing the connected components: if $\langle v_0, \dots, v_n, v_0 \rangle$ ($n \geq 2$) is a cycle, remove the edge $\{v_0, v_1\}$ from the graph.
Every walk using this edge can substitute $\langle v_1, \dots, v_n, v_0 \rangle$ (or the reverse path) for it.

Iteratively remove edges from $G$ in this way while preserving connectedness until this is no longer possible. The resulting graph $(V, E')$ is acyclic and connected and therefore a tree.

This implies $|E'| = |V| - 1$, but we also have $|E| = |V| - 1$.
This yields $|E| = |E'|$ and hence $E' = E$: the number of edges removable in this way must be 0. Hence $G$ is already acyclic. $\square$