Discrete Mathematics in Computer Science

B7. Sets: Countability

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B7.1 Countable Sets

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Comparing Cardinality

- Two sets A and B have the same cardinality if their elements can be paired (i.e. there is a bijection from A to B).
- Set A has a strictly smaller cardinality than set B if
  - we can map distinct elements of A to distinct elements of B (i.e. there is an injective function from A to B), and
  - $|A| \neq |B|$.

- This clearly makes sense for finite sets.
- What about infinite sets?
  Do they even have different cardinalities?

Countable and Countably Infinite Sets

Definition (countably infinite and countable)

A set A is countably infinite if $|A| = |\mathbb{N}_0|$.

A set A is countable if $|A| \leq |\mathbb{N}_0|$.

A set is countable if it is finite or countably infinite.

- We can count the elements of a countable set one at a time.
- The objects are "discrete" (in contrast to "continuous").
- Discrete mathematics deals with all kinds of countable sets.

Set of Even Numbers

- $\mathit{even} = \{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}$
- Obviously: $\mathit{even} \subset \mathbb{N}_0$
- Intuitively, there are twice as many natural numbers as even numbers — no?
- Is $|\mathit{even}| < |\mathbb{N}_0|$?

Set of Even Numbers

Theorem (set of even numbers is countably infinite)

The set of all even natural numbers is countably infinite, i. e. $|\{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}| = |\mathbb{N}_0|$.

Proof Sketch.

We can pair every even number $2n$ with natural number n. □

Set of Perfect Squares

Theorem (set of perfect squares is countably infinite)

The set of all perfect squares is countably infinite,
i. e. $|\{n^2 \mid n \in \mathbb{N}_0\}| = |\mathbb{N}_0|$.

Proof Sketch.

We can pair every square number n^2 with natural number n. □

Subsets of Countable Sets are Countable

In general:

Theorem (subsets of countable sets are countable)

Let A be a countable set. Every set B with $B \subseteq A$ is countable.

Proof.

Since A is countable there is an injective function f from A to $\mathbb{N}_0$.
The restriction of f to B is an injective function from B to $\mathbb{N}_0$. □

Set of the Positive Rationals

Theorem (set of positive rationals is countably infinite)

Set $\mathbb{Q}_+ = \{n \mid n \in \mathbb{Q} \text{ and } n > 0\} = \{p/q \mid p, q \in \mathbb{N}_1\}$ *is countably infinite.*

Proof idea.

$$
\begin{array}{ccccccccccc}
\frac{1}{1}\ (0) & \rightarrow & \frac{1}{2}\ (1) & & \frac{1}{3}\ (4) & \rightarrow & \frac{1}{4}\ (5) & & \frac{1}{5}\ (10) & \rightarrow & \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & & \\
\frac{2}{1}\ (2) & & \frac{2}{2}\ (\cdot) & & \frac{2}{3}\ (6) & & \frac{2}{4}\ (\cdot) & & \frac{2}{5} & & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & & & \\
\frac{3}{1}\ (3) & & \frac{3}{2}\ (7) & & \frac{3}{3}\ (\cdot) & & \frac{3}{4} & & \frac{3}{5} & & \cdots \\
 & \swarrow & & \nearrow & & & & & & & \\
\frac{4}{1}\ (8) & & \frac{4}{2}\ (\cdot) & & \frac{4}{3} & & \frac{4}{4} & & \frac{4}{5} & & \cdots \\
\downarrow & \nearrow & & & & & & & & & \\
\frac{5}{1}\ (9) & & \frac{5}{2} & & \frac{5}{3} & & \frac{5}{4} & & \frac{5}{5} & & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

$\square$

Union of Two Countable Sets is Countable

Theorem (union of two countable sets countable)

Let A and B be countable sets. Then $A \cup B$ is countable.

Proof sketch.

As A and B are countable there is an injective function f_A from A to $\mathbb{N}_0$, analogously f_B from B to $\mathbb{N}_0$.

We define function $f_{A\cup B}$ from $A \cup B$ to $\mathbb{N}_0$ as

$$f_{A\cup B}(e) = \begin{cases} 2f_A(e) & \text{if } e \in A \\ 2f_B(e) + 1 & \text{otherwise} \end{cases}$$

This $f_{A\cup B}$ is an injective function from $A \cup B$ to $\mathbb{N}_0$. □

Integers and Rationals

Theorem (sets of integers and rationals are countably infinite)

The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.

Without proof ($\rightsquigarrow$ exercises)

Union of More than Two Sets

Definition (arbitrary unions)

Let M be a set of sets. The union $\bigcup_{S \in M} S$ is the set with

$$x \in \bigcup_{S \in M} S \text{ iff exists } S \in M \text{ with } x \in S.$$

Countable Union of Countable Sets

Theorem

Let M be a countable set of countable sets.

Then $\bigcup_{S \in M} S$ is countable.

Proof sketch.

With $M = \{S_1, S_2, S_3, \dots\}$ (possibly finite) and each $S_i = \{x_{i1}, x_{i2}, \dots\}$ (possibly finite), we can use an analogous idea as for the countability of $\mathbb{Q}_+$ (skipping duplicates):

$$
\begin{array}{lcccccccccc}
S_1: & x_{11} & \rightarrow & x_{12} & & x_{13} & \rightarrow & x_{14} & & x_{15} & \rightarrow \\
 & & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \\
S_2: & x_{21} & & x_{22} & & x_{23} & & x_{24} & & x_{25} & \cdots \\
 & \downarrow & \nearrow & & \swarrow & & \nearrow & & & & \\
S_3: & x_{31} & & x_{32} & & x_{33} & & x_{34} & & x_{35} & \cdots \\
 & & \swarrow & & \nearrow & & & & & & \\
S_4: & x_{41} & & x_{42} & & x_{43} & & x_{44} & & x_{45} & \cdots \\
 & \downarrow & \nearrow & & & & & & & & \\
S_5: & x_{51} & & x_{52} & & x_{53} & & x_{54} & & x_{55} & \cdots \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

$\square$

Set of all Binary Trees is Countable

Theorem (set of all binary trees is countable)

The set $B = \{b \mid b \text{ is a binary tree}\}$ is countable.

Proof.

For $n \in \mathbb{N}_0$ the set B_n of all binary trees with n leaves is finite.

With $M = \{B_i \mid i \in \mathbb{N}_0\}$ the set of all binary trees is $B = \bigcup_{B' \in M} B'$.

Since M is a countable set of countable sets, B is countable. □

And Now?

We have seen several countably infinite sets.

What about our original questions?

- Do all infinite sets have the same cardinality?
- Are they all countably infinite?

Summary

- A set is countable if it has at most cardinality $|\mathbb{N}_0|$.
- If a set is countable and infinite, it is countably infinite.
- Sets $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.
- Every subset of a countable set is countable.
- Every countable union of countable sets is countable, in particular, the union of two countable sets is countable.