

Discrete Mathematics in Computer Science

B6. Sets: Comparing Cardinality and Hilbert's Hotel

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B6.1 Comparing Cardinality

B6.2 Hilbert's Hotel

B6.1 Comparing Cardinality

Finite Sets Revisited

We already know:

- ▶ The **cardinality** $|S|$ measures the size of set S .
- ▶ A set is **finite** if it has a finite number of elements.
- ▶ The **cardinality** of a finite set is the **number of elements** it contains.

A set is **infinite** if it has an infinite number of elements.

Do all infinite sets have the same cardinality?

Comparing the Cardinality of Sets

- ▶ Consider $A = \{1, 2\}$ and $B = \{\text{dog, cat, mouse}\}$.
- ▶ We can map distinct elements of A to distinct elements of B , e.g.

$$\begin{aligned} 1 &\mapsto \text{dog} \\ 2 &\mapsto \text{cat} \end{aligned}$$

- ▶ This is an **injective function** from A to B :
 - ▶ every element of A is mapped to an element of B ;
 - ▶ different elements of A are mapped to different elements of B .

Comparing Cardinality

Definition (cardinality not larger)

Set A has **cardinality less than or equal** to the cardinality of set B ($|A| \leq |B|$), if there is an **injective function** from A to B .

Comparing the Cardinality of Sets

- ▶ $A = \{1, 2, 3\}$ and $B = \{\text{dog, cat, mouse}\}$ have cardinality 3.
- ▶ We can pair their elements by a bijection from A to B :

$$\begin{aligned} 1 &\leftrightarrow \text{dog} \\ 2 &\leftrightarrow \text{cat} \\ 3 &\leftrightarrow \text{mouse} \end{aligned}$$

- ▶ This is a **bijection** from A to B .
 - ▶ Each element of A is paired with exactly one element of set B .
 - ▶ Each element of B is paired with exactly one element of A .
- ▶ If there is a bijection from A to B there is one from B to A (the inverse function).

Equinumerous Sets

We use the existence of a bijection also as criterion for infinite sets:

Definition (equinumerous sets)

Two sets A and B have the same cardinality ($|A| = |B|$) if there **exists a bijection** from A to B .

Such sets are called **equinumerous**.

Definition (strictly smaller cardinality)

Set A has **cardinality strictly less** than the cardinality of set B ($|A| < |B|$), if $|A| \leq |B|$ and $|A| \neq |B|$.

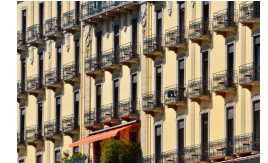
Consider set A and object $e \notin A$. Is $|A| < |A \cup \{e\}|$?

B6.2 Hilbert's Hotel

Hilbert's Hotel

Our intuition for finite sets does not always work for infinite sets.

- ▶ If in a hotel all rooms are occupied then it cannot accomodate additional guests.
- ▶ But **Hilbert's Grand Hotel** has **infinitely many rooms**.
- ▶ All these rooms are **occupied**.

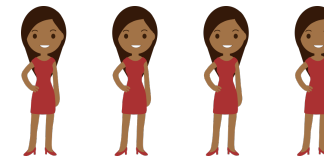


One More Guest Arrives



- ▶ Every guest moves from her current room n to room $n + 1$.
- ▶ Room 1 is then free.
- ▶ The new guest gets room 1.

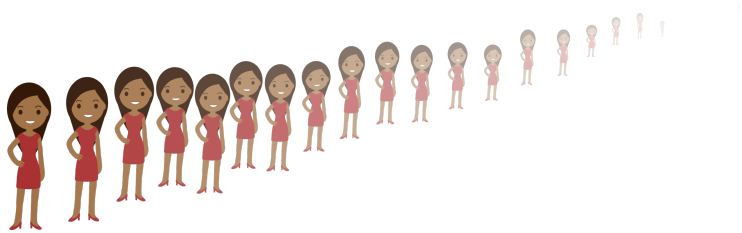
Four More Guests Arrive



- ▶ Every guest moves from her current room n to room $n + 4$.
- ▶ Rooms 1 to 4 are no longer occupied and can be used for the new guests.

→ Works for any finite number of additional guests.

An Infinite Number of Guests Arrives



- ▶ Every guest moves from her current room n to room $2n$.
- ▶ The infinitely many rooms with odd numbers are now available.
- ▶ The new guests fit into these rooms.

Can we Go further?

What if ...

- ▶ infinitely many coaches, each with an infinite number of guests
- ▶ infinitely many ferries, each with an infinite number of coaches, each with infinitely many guests
- ▶ ...

... arrive?

There are strategies for all these situations
as long as with “infinite” we mean “countably infinite”
and there is a finite number of layers.

Summary

- ▶ Set A has **cardinality less than or equal** the cardinality of set B ($|A| \leq |B|$), if **there is an injective function from A to B** .
- ▶ Sets A and B have the **same cardinality** ($|A| = |B|$) if **there exists a bijection from A to B** .
- ▶ Our intuition for finite sets does not always work for infinite sets.