

Discrete Mathematics in Computer Science

B5. Functions

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Partial and Total Functions

Important Building Blocks of Discrete Mathematics

Important building blocks:

- sets
- relations
- functions

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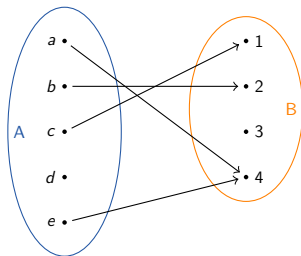
In principle, functions are just a special kind of relations:

- $f : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ with $f(x) = x^2$
- relation R over $\mathbb{N}_0$ with $R = \{(x, x^2) \mid x \in \mathbb{N}_0\}$.

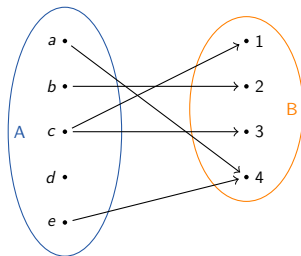
Functional Relations

Definition

A binary relation R over sets A and B is **functional** if **for every $a \in A$ there is at most one $b \in B$ with $(a, b) \in R$.**



functional



not functional

Functions – Examples

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■ $distance : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ with

$$distance((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Partial Function – Example

Partial function $r : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q}$ with

$$r(n, d) = \begin{cases} \frac{n}{d} & \text{if } d \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Partial Functions

Definition (Partial function)

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If there is no $y \in B$ with $(x, y) \in G$, then **$f(x)$ is undefined**.

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has graph $\{((n, d), \frac{n}{d}) \mid n \in \mathbb{Z}, d \in \mathbb{Z} \setminus \{0\}\} \subseteq \mathbb{Z}^2 \times \mathbb{Q}$.

Domain (of Definition), Codomain, Image

Definition (Domain of definition, codomain, image)

Let $f : A \rightarrow B$ be a partial function.

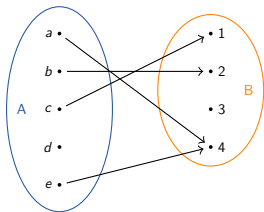
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$$f : \{a, b, c, d, e\} \rightarrowtail \{1, 2, 3, 4\}$$

$$f(a) = 4, f(b) = 2, f(c) = 1, f(e) = 4$$

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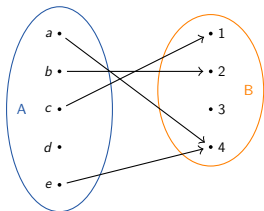
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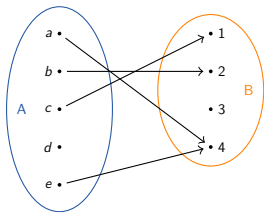
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The **image** (or **range**) of f is the set

$$\text{img}(f) = \{y \mid \text{there is an } x \in A \text{ with } f(x) = y\}.$$



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Preimage

The preimage contains all elements of the domain that are mapped to given elements of the codomain.

Definition (Preimage)

Let $f : A \rightarrow B$ be a partial function and let $Y \subseteq B$.

The **preimage of Y under f** is the set

$$f^{-1}[Y] = \{x \in A \mid f(x) \in Y\}.$$

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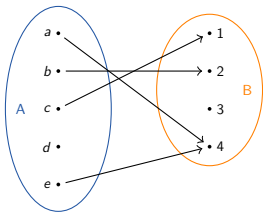
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$$f^{-1}[\{1, 2\}] =$$

Total Functions

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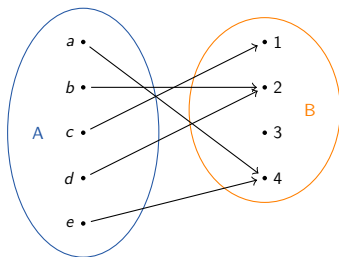
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Specifying a Function

Some common ways of specifying a function:

- Listing the mapping **explicitly**, e. g.

$$f(a) = 4, f(b) = 2, f(c) = 1, f(e) = 4 \text{ or}$$

$$f = \{a \mapsto 4, b \mapsto 2, c \mapsto 1, e \mapsto 4\}$$

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 $0! = 1$ and
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- In terms of other functions, e. g. inverse, composition

Relationship to Functions in Programming

```
def factorial(n):  
    if n == 0:  
        return 1  
    else:  
        return n * factorial(n-1)
```

→ Relationship between recursion and recurrence

Relationship to Functions in Programming

```
def foo(n):  
    value = ...  
    while <some condition>:  
        ...  
        value = ...  
    return value
```

- Does possibly not terminate on all inputs.
- Value is undefined for such inputs.
- Theoretical computer science: partial function

Relationship to Functions in Programming

```
import random
counter = 0

def bar(n):
    print("Hi! I got input", n)
    global counter
    counter += 1
    return random.choice([1,2,n])
```

- Functions in programming don't always compute mathematical functions (except *purely functional languages*).
- In addition, not all mathematical functions are computable.

Questions



Questions?

Operations on Partial Functions

Restrictions and Extensions

Definition (Restriction and extension)

Let $f : A \rightarrowtail B$ be a partial function and let $X \subseteq A$.

The **restriction of f to X** is the partial function $f|_X : X \rightarrowtail B$ with $f|_X(x) = f(x)$ for all $x \in X$.

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What's the restriction of f to its domain?

Function Composition

Definition (Composition of partial functions)

Let $f : A \rightharpoonup B$ and $g : B \rightharpoonup C$ be partial functions.

The **composition of f and g** is $g \circ f : A \rightharpoonup C$ with

$$(g \circ f)(x) = \begin{cases} g(f(x)) & \text{if } f \text{ is defined for } x \text{ and} \\ & g \text{ is defined for } f(x) \\ \text{undefined} & \text{otherwise} \end{cases}$$

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Example:

$$f : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \quad \text{with } f(x) = x^2$$

$$g : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \quad \text{with } g(x) = x + 3$$

$$(g \circ f)(x) =$$

Properties of Function Composition

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- $(f \circ g)(x) = (x + 3)^2$

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- $(g \circ f)(x) = x^2 + 3$
- $(f \circ g)(x) = (x + 3)^2$

- **associative**, i. e. $h \circ (g \circ f) = (h \circ g) \circ f$

→ analogous to associativity of relation composition

Function Composition in Programming

We implicitly compose functions all the time...

```
def foo(n):  
    ...  
    x = somefunction(n)  
    y = someotherfunction(x)  
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Many languages also allow explicit composition of functions, e. g. in Haskell:

```
incr x = x + 1  
square x = x * x  
squareplusone = incr . square
```

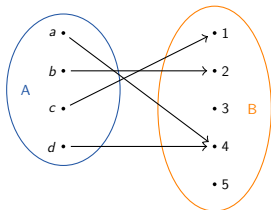

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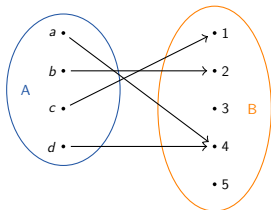
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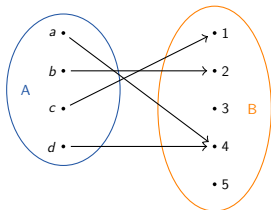
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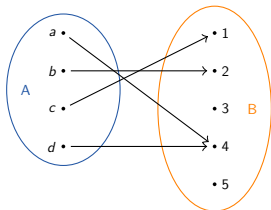
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- There can be values of the codomain that aren't the image of any element of the domain.

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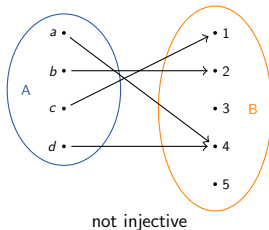
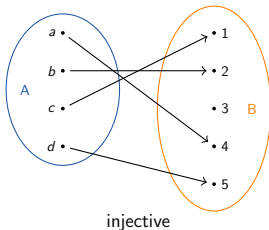
- Partial functions map every element of their domain to at most one element of their codomain, total functions map it to exactly one such value.
- Different elements of the domain can have the same image.
- There can be values of the codomain that aren't the image of any element of the domain.
- We often want to exclude such cases
→ define additional properties to say this quickly

Injective Functions

An **injective function** maps distinct elements of its domain to distinct elements of its co-domain.

Definition (Injective function)

A function $f : A \rightarrow B$ is **injective** (also **one-to-one** or an **injection**) if for all $x, y \in A$ with $x \neq y$ it holds that $f(x) \neq f(y)$.



Injective Functions – Examples

Which of these functions are injective?

- $f : \mathbb{Z} \rightarrow \mathbb{N}_0$ with $f(x) = |x|$
- $g : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ with $g(x) = x^2$
- $h : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ with $h(x) = \begin{cases} x - 1 & \text{if } x \text{ is odd} \\ x + 1 & \text{if } x \text{ is even} \end{cases}$

Composition of Injective Functions

Theorem

If $f : A \rightarrow B$ and $g : B \rightarrow C$ are injective functions then also $g \circ f$ is injective.

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If $f : A \rightarrow B$ and $g : B \rightarrow C$ are injective functions then also $g \circ f$ is injective.

Proof.

Consider arbitrary elements $x, y \in A$ with $x \neq y$.

Since f is injective, we know that $f(x) \neq f(y)$.

As g is injective, this implies that $g(f(x)) \neq g(f(y))$.

With the definition of $g \circ f$, we conclude that

$(g \circ f)(x) \neq (g \circ f)(y)$.

Overall, this shows that $g \circ f$ is injective.

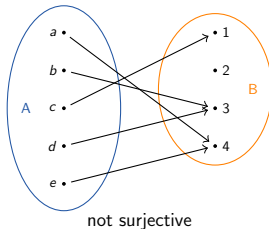
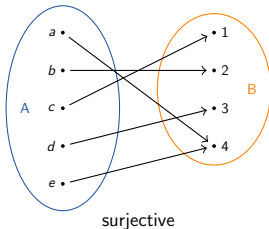


Surjective Functions

A **surjective function** maps at least one elements to every element of its co-domain.

Definition (Surjective function)

A function $f : A \rightarrow B$ is **surjective** (also **onto** or a **surjection**) if its **image is equal to its codomain**,
i. e. for all $y \in B$ there is an $x \in A$ with $f(x) = y$.



Surjective Functions – Examples

Which of these functions are surjective?

- $f : \mathbb{Z} \rightarrow \mathbb{N}_0$ with $f(x) = |x|$
- $g : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ with $g(x) = x^2$
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Proof.

Consider an arbitrary element $z \in C$.

Since g is surjective, there is a $y \in B$ with $g(y) = z$.

As f is surjective, for such a y there is an $x \in A$ with $f(x) = y$ and thus $g(f(x)) = z$.

Overall, for every $z \in C$ there is an $x \in A$ with $(g \circ f)(x) = g(f(x)) = z$, so $g \circ f$ is surjective.



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Bijjective Functions

A **bijjective function** pairs every element of its domain with exactly one element of its codomain and every element of the codomain is paired with exactly one element of the domain.

Definition (Bijjective function)

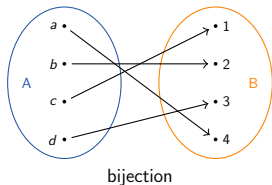
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Bijjective Functions

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A function is **bijjective** (also a **one-to-one correspondence** or a **bijection**) if it is **injective** and **surjective**.

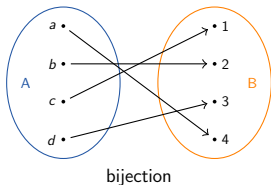


Bijjective Functions

A **bijjective function** pairs every element of its domain with exactly one element of its codomain and every element of the codomain is paired with exactly one element of the domain.

Definition (Bijjective function)

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Corollary

The composition of two bijective functions is bijective.

Bijjective Functions – Examples

Which of these functions are bijective?

- $f : \mathbb{Z} \rightarrow \mathbb{N}_0$ with $f(x) = |x|$
- $g : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ with $g(x) = x^2$
- $h : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ with $h(x) = \begin{cases} x - 1 & \text{if } x \text{ is odd} \\ x + 1 & \text{if } x \text{ is even} \end{cases}$

Inverse Function

Definition

Let $f : A \rightarrow B$ be a bijection.

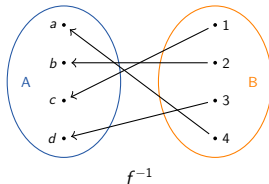
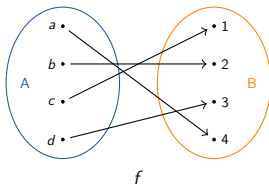
The **inverse function** of f is the function $f^{-1} : B \rightarrow A$ with $f^{-1}(y) = x$ iff $f(x) = y$.

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Inverse Function and Composition

Theorem

Let $f : A \rightarrow B$ be a bijection.

- ① *For all $x \in A$ it holds that $f^{-1}(f(x)) = x$.*
- ② *For all $y \in B$ it holds that $f(f^{-1}(y)) = y$.*
- ③ *f^{-1} is a bijection from B to A .*
- ④ *$(f^{-1})^{-1} = f$*

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Proof sketch.

- 1 For $x \in A$ let $y = f(x)$. Then $f^{-1}(f(x)) = f^{-1}(y) = x$
- 2 For $y \in B$ there is exactly one x with $y = f(x)$. With this x it holds that $f^{-1}(y) = x$ and overall $f(f^{-1}(y)) = f(x) = y$.

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- 3 Surjective: for all $x \in A$, f^{-1} maps $f(x)$ to x (cf. (1)).
Injective: if $f^{-1}(y) = f^{-1}(y')$ then $f(f^{-1}(y)) = f(f^{-1}(y'))$,
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so with (2) we have $y = y'$.
- 4 Def. of inverse: $(f^{-1})^{-1}(x) = y$ iff $f^{-1}(y) = x$ iff $f(x) = y$.

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Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be bijections.

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Proof.

We need to show that for all $x \in C$ it holds that

$$(g \circ f)^{-1}(x) = (f^{-1} \circ g^{-1})(x).$$

Consider an arbitrary $x \in C$ and let $y = (g \circ f)^{-1}(x)$.

By the definition of the inverse $(g \circ f)(y) = g(f(y)) = x$.

Let $z = f(y)$.

From $x = g(f(y))$, we know that $x = g(z)$ and thus $g^{-1}(x) = z$.

From $z = f(y)$ we get $f^{-1}(z) = y$.

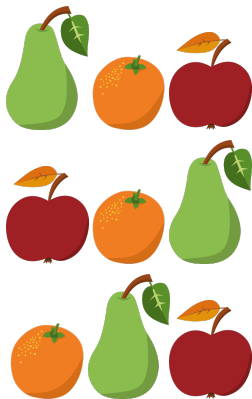
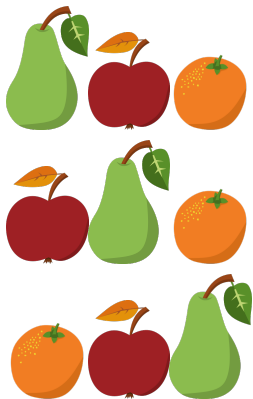
This gives $(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x)) = f^{-1}(z) = y$. □

Questions



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Permutations



Permutation – Definition

Definition (Permutation)

Let S be a set. A **bijection** $\pi : S \rightarrow S$ is called a **permutation of S** .

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The inverse of a permutation is again a permutation.

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 - The object at position 1 was moved to position 4,
 - the one from position 3 to position 1,
 - the one from position 4 to position 3 and
 - the one at position 2 stayed where it was.
- This corresponds to the permutation
 $\sigma : \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$ with
 $\sigma(1) = 4, \sigma(2) = 2, \sigma(3) = 1, \sigma(4) = 3$

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Determine the arrangement of some objects after applying a permutation that operates on the locations.

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Then $\pi \circ f$ describes the resulting configuration.

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For example $g(1) = \text{pear}$, $g(2) = \text{apple}$, $g(3) = \text{orange}$ for .

Then $(f^{-1} \circ g)(1) = \text{apple}$, $(f^{-1} \circ g)(2) = \text{pear}$, $(f^{-1} \circ g)(3) = \text{orange}$
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Then $g \circ f^{-1}$ describes the permutation of locations.

Questions



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Summary

- **injective function**: maps distinct elements of its domain to distinct elements of its co-domain.
- **surjective function**: maps at least one element to every element of its co-domain.
- **bijective function**: injective and surjective
→ one-to-one correspondence
- Bijective functions are invertible. The **inverse** function of f maps the image of x under f to x .
- **Permutations** are bijections from a set to itself. They can be used to describe rearrangements of objects.