

Discrete Mathematics in Computer Science

B2. Relations

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Relations

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- Intuitively, a mathematical relation connects elements from several (possibly different) sets by specifying related groupings.
- We already know some relations, e. g.
 - $\subseteq$ relation for sets
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- There are also relations of **higher arity**, e. g.
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- There are also relations of **higher arity**, e. g.
 - " $x + y = z$ " for integers x, y, z .
 - "The name, address and office number belong to the same person."
- Relations are for example important for relational databases, semantic networks or knowledge representation and reasoning.

Relations

Definition (Relation)

Let $S_1, \dots, S_n$ be sets.

A **relation over $S_1, \dots, S_n$** is a set $R \subseteq S_1 \times \dots \times S_n$.

The **arity** of R is n .

A relation of arity n is a set of n -tuples.

German: Relation, Stelligkeit

Relations: Examples

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- $R = \{(x, y, z) \mid x, y, z \in \mathbb{Z} \text{ and } x + y = z\}$
- $R' = \{(\text{Gabi Röger, Spiegelgasse 1, 04.005}),$
 $(\text{Malte Helmert, Spiegelgasse 1, 06.004}),$
 $(\text{David Speck, Spiegelgasse 5, 04.003})\}$

Questions



Questions?

Properties of Binary Relations

Binary Relation

A binary relation is a relation of arity 2:

Definition (binary relation)

A **binary relation** is a relation over two sets A and B .

German: zweistellige Relation, homogene Relation

Binary Relation

A binary relation is a relation of arity 2:

Definition (binary relation)

A **binary relation** is a relation over two sets A and B .

- Instead of $(x, y) \in R$, we also write xRy , e. g.
 $x \leq y$ instead of $(x, y) \in \leq$
- If the sets are equal, we say “ R is a binary relation over A ” instead of “ R is a binary relation over A and A ”.
- Such a relation over a set is also called a **homogeneous relation** or an **endorelation**.

German: zweistellige Relation, homogene Relation

Reflexivity

A **reflexive** relation relates every object to itself.

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Definition (symmetric)

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How do these properties relate to irreflexivity?

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- A **binary relation** is a relation over two sets.
- A binary relation over set S is a relation $R \subseteq S \times S$ and also called a **homogeneous relation**.
- A binary relation R over A is
 - **reflexive** if $(a, a) \in R$ for all $a \in A$,
 - **irreflexive** if $(a, a) \notin R$ for all $a \in A$,
 - **symmetric** if for all $a, b \in A$ it holds that $(a, b) \in R$ iff $(b, a) \in R$,
 - **asymmetric** if for all $a, b \in A$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$,
 - **antisymmetric** if for all $a, b \in A$ with $a \neq b$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$,
 - **transitive** if for all $a, b, c \in A$ it holds that if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Special Classes of Relations

- Some important classes of relations are defined in terms of these properties.
 - **Equivalence relation:** reflexive, symmetric, transitive
 - **Partial order:** reflexive, antisymmetric, transitive
 - **Strict order:** irreflexive, asymmetric, transitive
 - ...
- We will consider these and other classes in detail.