

Discrete Mathematics in Computer Science

B2. Relations

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B2.1 Relations

B2.2 Properties of Binary Relations

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Relations: Informally

- ▶ Intuitively, a mathematical relation connects elements from several (possibly different) sets by specifying related groupings.
- ▶ We already know some relations, e. g.
 - ▶ $\subseteq$ relation for sets
 - ▶ $\leq$ relation for natural numbers
- ▶ These are examples of **binary** relations, considering **pairs of objects**.
- ▶ There are also relations of **higher arity**, e. g.
 - ▶ “ $x + y = z$ ” for integers x, y, z .
 - ▶ “The name, address and office number belong to the same person.”
- ▶ Relations are for example important for relational databases, semantic networks or knowledge representation and reasoning.

Relations

Definition (Relation)

Let $S_1, \dots, S_n$ be sets.

A **relation over** $S_1, \dots, S_n$ is a set $R \subseteq S_1 \times \dots \times S_n$.

The **arity** of R is n .

A relation of arity n is a set of n -tuples.

German: Relation, Stelligkeit

Relations: Examples

- ▶ $\subseteq = \{(S, S') \mid S \text{ and } S' \text{ are sets and for every } x \in S \text{ it holds that } x \in S'\}$
- ▶ $\leq = \{(x, y) \mid x, y \in \mathbb{N}_0 \text{ and } x < y \text{ or } x = y\}$
- ▶ $R = \{(x, y, z) \mid x, y, z \in \mathbb{Z} \text{ and } x + y = z\}$
- ▶ $R' = \{(\text{Gabi Röger, Spiegelgasse 1, 04.005}),$
 $(\text{Malte Helmert, Spiegelgasse 1, 06.004}),$
 $(\text{David Speck, Spiegelgasse 5, 04.003})\}$

B2.2 Properties of Binary Relations

Binary Relation

A binary relation is a relation of arity 2:

Definition (binary relation)

A **binary relation** is a relation over two sets A and B .

- ▶ Instead of $(x, y) \in R$, we also write xRy , e. g.
 $x \leq y$ instead of $(x, y) \in \leq$
- ▶ If the sets are equal, we say “ R is a binary relation over A ” instead of “ R is a binary relation over A and A ”.
- ▶ Such a relation over a set is also called a **homogeneous relation** or an **endorelation**.

German: zweistellige Relation, homogene Relation

Reflexivity

A **reflexive** relation relates every object to itself.

Definition (reflexive)

A binary relation R over set A is **reflexive** if **for all $a \in A$ it holds that $(a, a) \in R$.**

Which of these relations are reflexive?

- ▶ $R = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ equality relation $=$ on natural numbers
- ▶ less-than relation $\leq$ on natural numbers
- ▶ strictly-less-than relation $<$ on natural numbers

German: reflexiv

Irreflexivity

A **irreflexive** relation never relates an object to itself.

Definition (irreflexive)

A binary relation R over set A is **irreflexive** if **for all $a \in A$ it holds that $(a, a) \notin R$.**

Which of these relations are irreflexive?

- ▶ $R = \{(a, a), (a, b), (a, c), (b, a), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ equality relation $=$ on natural numbers
- ▶ less-than relation $\leq$ on natural numbers
- ▶ strictly-less-than relation $<$ on natural numbers

German: irreflexiv

Symmetry

Definition (symmetric)

A binary relation R over set A is **symmetric** if **for all $a, b \in A$ it holds that $(a, b) \in R$ iff $(b, a) \in R$.**

Which of these relations are symmetric?

- ▶ $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$ over $\{a, b, c\}$
- ▶ $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ equality relation $=$ on natural numbers
- ▶ less-than relation $\leq$ on natural numbers
- ▶ strictly-less-than relation $<$ on natural numbers

German: symmetrisch

Asymmetry and Antisymmetry

How do these properties relate to irreflexivity?

Definition (asymmetric and antisymmetric)

Let R be a binary relation over set A .

Relation R is **asymmetric** if

for all $a, b \in A$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$.

Relation R is **antisymmetric** if for all $a, b \in A$ with $a \neq b$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$.

Which of these relations are asymmetric/antisymmetric?

- ▶ $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$ over $\{a, b, c\}$
- ▶ $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ equality relation $=$ on natural numbers
- ▶ less-than relation $\leq$ on natural numbers
- ▶ strictly-less-than relation $<$ on natural numbers

German: asymmetrisch, antisymmetrisch

Transitivity

Definition

A binary relation R over set A is **transitive** if it holds for all $a, b, c \in A$ that
if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Which of these relations are transitive?

- ▶ $R = \{(a, a), (a, b), (a, c), (b, a), (c, a), (c, c)\}$ over $\{a, b, c\}$
- ▶ $R = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, c)\}$ over $\{a, b, c\}$
- ▶ equality relation $=$ on natural numbers
- ▶ less-than relation $\leq$ on natural numbers
- ▶ strictly-less-than relation $<$ on natural numbers

German: transitiv

Summary

- ▶ A **relation** over sets $S_1, \dots, S_n$ is a set $R \subseteq S_1 \times \dots \times S_n$.
- ▶ A **binary relation** is a relation over two sets.
- ▶ A binary relation over set S is a relation $R \subseteq S \times S$ and also called a **homogeneous relation**.
- ▶ A binary relation R over A is
 - ▶ **reflexive** if $(a, a) \in R$ for all $a \in A$,
 - ▶ **irreflexive** if $(a, a) \notin R$ for all $a \in A$,
 - ▶ **symmetric** if for all $a, b \in A$ it holds that $(a, b) \in R$ iff $(b, a) \in R$,
 - ▶ **asymmetric** if for all $a, b \in A$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$,
 - ▶ **antisymmetric** if for all $a, b \in A$ with $a \neq b$ it holds that if $(a, b) \in R$ then $(b, a) \notin R$,
 - ▶ **transitive** if for all $a, b, c \in A$ it holds that if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Special Classes of Relations

- ▶ Some important classes of relations are defined in terms of these properties.
 - ▶ **Equivalence relation**: reflexive, symmetric, transitive
 - ▶ **Partial order**: reflexive, antisymmetric, transitive
 - ▶ **Strict order**: irreflexive, asymmetric, transitive
 - ▶ ...
- ▶ We will consider these and other classes in detail.