

# Discrete Mathematics in Computer Science

## A1. Organizational Matters

Malte Helmert, Gabriele Röger

University of Basel

September 17, 2025

# Organizational Matters

# People

## Lecturers



Malte Helmert

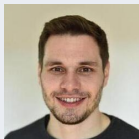
- email: `malte.helmert@unibas.ch`
- office: room 06.004, Spiegelgasse 1



Gabi Röger

- email: `gabriele.roeger@unibas.ch`
- office: room 04.005, Spiegelgasse 1

## Assistant



David Speck

- email: `davidjakob.speck@unibas.ch`
- office: room 04.003, Spiegelgasse 5

# People



## Tutors

- Maria Desteffani ([maria.desteffani@unibas.ch](mailto:maria.desteffani@unibas.ch))
- Pascal von Fellenberg ([pascal.vonfellenberg@unibas.ch](mailto:pascal.vonfellenberg@unibas.ch))
- Carina Schrenk ([carina.schrenk@unibas.ch](mailto:carina.schrenk@unibas.ch))
- Carina Fehr ([carina.fehr@unibas.ch](mailto:carina.fehr@unibas.ch))



# Target Audience

## target audience:

- this is an introductory course on the Bachelor's level
- we cover mathematical foundations that are particularly useful for the computer science curriculum
- main target audience: B.Sc. Computer Science, 1st semester
- all other students welcome

# Enrolment

- `https://services.unibas.ch/`
- **official deadline:** October 13
- better today, so that you get all relevant emails and access to the ADAM workspace

# Discrete Mathematics Course on ADAM

## ADAM

<https://adam.unibas.ch/>

- link to website with slides
- submission of exercise sheets
- model solutions for exercise sheets
- link to Discord server (for interaction among participants, but you also get answers from lecturers, assistant and tutors)
- additional material

# Language

- The course is taught in English.
- All lecture material is in English.
- We (lecturers, assistant, tutors) speak German and English.
- You are also welcome to ask questions in German.
- Also exercise submissions can be in English or German.

# Lectures

- Mon 16:15–18:00, Hörsaal U1.131, Biozentrum  
Wed 16:15–17:00, Hörsaal 1, Pharmazentrum
- first half of the course taught by Gabi Röger,  
second half by Malte Helmert
- on December 17: Q&A session for exam preparation

# Exercises

## Exercise sheets (homework assignments):

- mostly theoretical exercises
- exercise sheets on ADAM every Monday after the lecture
- must be solved in **groups of two or three**  
(not alone or in larger groups)
- due on the following Sunday (23:59)  
(upload to ADAM at <https://adam.unibas.ch/>)
- we only accept readable PDFs  
→ with a bonus point per sheet created with  $\text{\LaTeX}$   
(template, cheat sheet and intro on ADAM)

Question: Who has experience with  $\text{\LaTeX}$ ?

# Exercise Sessions With Tutors

## Exercise Sessions (starting September 24/25/27)

Wed 17:15–18:00	Alte Universität, Seminarraum –201 with Carina S.
Wed 17:15–18:00	Spiegelgasse 1, Computer-Labor U1.001 with Pascal
Thu 17:15–18:00	Spiegelgasse 1, Seminarraum 00.003 with Maria
Fri 17:15–18:00	Pharmazentrum, Labor U1075 with Carina F.

- common mistakes/misconceptions  
(full model solutions on ADAM)
- questions about exercise sheets and the course
- as time permits, support while you solve the exercises

**important:** please fill in the survey on ADAM for the group allocation until **Friday 12:00** (September 19).

# Exam

- Written exam
- 6 ECTS credits
- Monday, January 19, 2026, 16:00-18:00
- Maurice E. Müller Saal, Biozentrum
- admission to exam: 50% of the exercise marks
- grade for course determined exclusively by the exam



# Required Time

## Official calculation

- 1 CP  $\approx$  30 hours
- The course has 6 CP.
- You need to invest about 180 hours.
- With 40 hours for exam preparation, this leaves 10–11 hours/week during the teaching period.

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## Alternative calculation

- A full-time student achieves 30 CP per semester.
- The course corresponds to 1/5 of 30 CP.
- With a 42h week, this still corresponds to 8.4 hours/week.

# Plagiarism

## Plagiarism

Plagiarism is presenting someone else's work, ideas, or words as your own, without proper attribution.

For example:

- Using someone's text without citation
- Paraphrasing too closely
- Using information from a source without attribution
- Passing off AI-generated content as your own original work

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Long-term impact:

- You undermine your own learning.
- You start to lose confidence in your ability to think, write, and solve problems independently.
- Damage to academic reputation and professional consequences in future careers

# Plagiarism in Exercises

- You may discuss material from the course, including the exercise assignments, with your peers.
- **But:** You have to independently write down your exercise solutions (in your team).
- Help from an LLM is acceptable to the same extent as it is acceptable from someone who is not a member of your team.

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- 0 marks for the exercise sheet (first time)
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**If in doubt:** check with us what is (and isn't) OK **before submitting**  
**Exercises too difficult?** We are happy to help!

# Special Needs?

- We (and the university) strive for equality of students with disabilities or chronic illnesses.
- Contact the lecturers for small adaptations.
- Contact the Students Without Barriers (StoB) service point for general adaptations and disadvantage compensation.



# Questions on Organization



Questions?

# About this Course

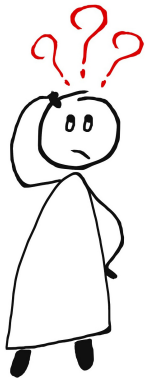
# Content: Discrete Mathematics in Computer Science

- mathematical thinking and proof techniques
- sets and relations
- group theory and permutations
- modular arithmetic
- graphs and trees
- formal logic

# Learning Goals

- proficiency in abstract thinking
- ability to formalize mathematical ideas and arguments
- knowledge of common mathematical tools in computer science

# Questions about the Course



Questions?

# Discrete Mathematics in Computer Science

## A2. Sets: Foundations

Malte Helmert, Gabriele Röger

University of Basel

September 22, 2025

# Sets

# Important Building Blocks of Discrete Mathematics

- sets
- relations
- functions

These topics will mainly be the content of part B of the course.



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- sets
- relations
- functions

These topics will mainly be the content of part B of the course.  
We cover some foundations on sets already now because we will use them for illustrating proof techniques.

# Sets

## Definition

A **set** is an **unordered collection** of **distinct** objects.

German: Menge

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- **unordered**: no notion of a “first” or “second” object,  
e. g.  $\{Alice, Bob, Charly\} = \{Charly, Bob, Alice\}$

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- **unordered**: no notion of a “first” or “second” object,  
e. g.  $\{Alice, Bob, Charly\} = \{Charly, Bob, Alice\}$
- **distinct**: each object contained **at most once**,  
e. g.  $\{Alice, Bob, Charly\} = \{Alice, Charly, Bob, Alice\}$

German: Menge

# Notation

## ■ Specification of sets

- **explicit**, listing all elements, e. g.  $A = \{1, 2, 3\}$
- **implicit** with **set-builder notation**,  
specifying a **property** characterizing all elements,  
e. g.  $A = \{x \mid x \in \mathbb{N}_0 \text{ and } 1 \leq x \leq 3\}$ ,  
 $B = \{n^2 \mid n \in \mathbb{N}_0\}$
- **implicit**, as a **sequence with dots**,  
e. g.  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
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**Question:** Is it true that  $1 \in \{\{1, 2\}, 3\}$ ?

German: Element, leere Menge



# Special Sets

- **Natural numbers**  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$

German: Natürliche ( $\mathbb{N}_0$ ), ganze ( $\mathbb{Z}$ ), rationale ( $\mathbb{Q}$ ), reelle ( $\mathbb{R}$ ) Zahlen

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- **Rational numbers**  $\mathbb{Q} = \{n/d \mid n \in \mathbb{Z}, d \in \mathbb{N}_1\}$

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- **Rational numbers**  $\mathbb{Q} = \{n/d \mid n \in \mathbb{Z}, d \in \mathbb{N}_1\}$
- **Real numbers**  $\mathbb{R} = (-\infty, \infty)$

Why do we use interval notation?

Why didn't we introduce it before?

German: Natürliche ( $\mathbb{N}_0$ ), ganze ( $\mathbb{Z}$ ), rationale ( $\mathbb{Q}$ ), reelle ( $\mathbb{R}$ ) Zahlen

# Questions



Questions?

# Russell's Paradox

# Excursus: Barber Paradox

## Barber Paradox

In a town there is only one barber, who is male.  
The barber shaves all men in the town,  
and only those, who do not shave themselves.





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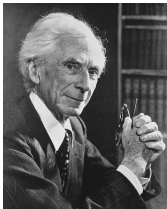
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Who shaves the barber?



We can exploit the self-reference to derive a contradiction.

# Russell's Paradox

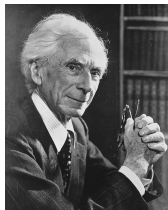


Bertrand Russell

## Question

Is the collection of all sets that do not contain themselves as a member a set?

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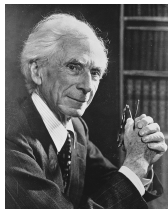
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Is the collection of all sets that do not contain themselves as a member a set?

Is  $S = \{M \mid M \text{ is a set and } M \notin M\}$  a set?

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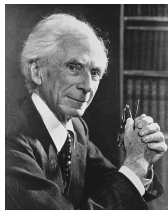
Assume that  $S$  is a set.

If  $S \notin S$  then  $S \in S \rightsquigarrow$  Contradiction

If  $S \in S$  then  $S \notin S \rightsquigarrow$  Contradiction

Hence, there is no such set  $S$ .

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Is the collection of all sets that do not contain themselves as a member a set?

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Hence, there is no such set  $S$ .

→ Not every property used in set-builder notation defines a set.

# Questions



Questions?

# Relations on Sets



# Equality

## Definition (Axiom of Extensionality)

Two sets  $A$  and  $B$  are **equal** (written  $A = B$ ) if every element of  $A$  is an element of  $B$  and vice versa.

Two sets are equal if they contain the same elements.

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Two sets are equal if they contain the same elements.

We write  $A \neq B$  to indicate that  $A$  and  $B$  are **not** equal.

# Subsets and Supersets

- $A \subseteq B$ :  $A$  is a **subset** of  $B$ ,  
i. e., every element of  $A$  is an element of  $B$
- $A \subset B$ :  $A$  is a **strict subset** of  $B$ ,  
i. e.,  $A \subseteq B$  and  $A \neq B$ .
- $A \supseteq B$ :  $A$  is a **superset** of  $B$  if  $B \subseteq A$ .
- $A \supset B$ :  $A$  is a **strict superset** of  $B$  if  $B \subset A$ .

German: Teilmenge, echte Teilmenge, Obermenge, echte Obermenge

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We write  $A \not\subseteq B$  to indicate that  $A$  is **not** a subset of  $B$ .

Analogously:  $\not\subset$ ,  $\not\supseteq$ ,  $\not\supset$

German: Teilmenge, echte Teilmenge, Obermenge, echte Obermenge

# Power Set

## Definition (Power Set)

The **power set**  $\mathcal{P}(S)$  of a set  $S$  is the set of all subsets of  $S$ .  
That is,

$$\mathcal{P}(S) = \{M \mid M \subseteq S\}.$$

Example:  $\mathcal{P}(\{a, b\}) =$

German: Potenzmenge

# Questions



Questions?

# Set Operations

## Set Operations

Set operations allow us to express sets in terms of other sets



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- **intersection**  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



If  $A \cap B = \emptyset$  then  $A$  and  $B$  are **disjoint**.

German: Schnitt, disjunkt

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- **set difference**  $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



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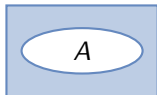
- **union**  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



- **set difference**  $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



- **complement**  $\bar{A} = B \setminus A$ , where  $A \subseteq B$  and  $B$  is the set of all considered objects (in a given context)



German: Schnitt, disjunkt, Vereinigung,  
Differenz, Komplement

# Properties of Set Operations: Commutativity

## Theorem (Commutativity of $\cup$ and $\cap$ )

*For all sets  $A$  and  $B$  it holds that*

- $A \cup B = B \cup A$  and
- $A \cap B = B \cap A$ .

German: Kommutativität

# Properties of Set Operations: Commutativity

## Theorem (Commutativity of $\cup$ and $\cap$ )

*For all sets  $A$  and  $B$  it holds that*

- $A \cup B = B \cup A$  and
- $A \cap B = B \cap A$ .

**Question:** Is the set difference also commutative,  
i. e. is  $A \setminus B = B \setminus A$  for all sets  $A$  and  $B$ ?

German: Kommutativität

# Properties of Set Operations: Associativity

## Theorem (Associativity of $\cup$ and $\cap$ )

*For all sets  $A, B$  and  $C$  it holds that*

- $(A \cup B) \cup C = A \cup (B \cup C)$  and
- $(A \cap B) \cap C = A \cap (B \cap C)$ .

German: Assoziativität

# Properties of Set Operations: Distributivity

Theorem (Union distributes over intersection and vice versa)

*For all sets  $A, B$  and  $C$  it holds that*

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$  and
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .

German: Distributivität



# Properties of Set Operations: De Morgan's Law



Augustus De Morgan

British mathematician (1806-1871)

## Theorem (De Morgan's Law)

*For all sets  $A$  and  $B$  it holds that*

- $\overline{A \cup B} = \overline{A} \cap \overline{B}$  and
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$ .

# Questions



Questions?

# Cardinality of Finite Sets

# Cardinality of Sets

The **cardinality**  $|S|$  measures the size of set  $S$ .

A set is **finite** if it has a finite number of elements.

## Definition (Cardinality)

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German: Kardinalität oder Mächtigkeit

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## Definition (Cardinality)

The **cardinality** of a finite set is the **number of elements** it contains.

- $|\emptyset| =$
- $|\{x \mid x \in \mathbb{N}_0 \text{ and } 2 \leq x < 5\}| =$
- $|\{3, 0, \{1, 3\}\}| =$
- $|\mathcal{P}(\{1, 2\})| =$

German: Kardinalität oder Mächtigkeit

# Cardinality of the Union of Sets

## Theorem

*For finite sets  $A$  and  $B$  it holds that  $|A \cup B| = |A| + |B| - |A \cap B|$ .*

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## Theorem

For finite sets  $A$  and  $B$  it holds that  $|A \cup B| = |A| + |B| - |A \cap B|$ .

## Corollary

If finite sets  $A$  and  $B$  are *disjoint* then  $|A \cup B| = |A| + |B|$ .

# Cardinality of the Power Set

## Theorem

Let  $S$  be a finite set. Then  $|\mathcal{P}(S)| = 2^{|S|}$ .

## Proof sketch.

We can construct a subset  $S'$  by iterating over all elements  $e$  of  $S$  and deciding whether  $e$  becomes a member of  $S'$  or not.

We make  $|S|$  independent decisions, each between two options. Hence, there are  $2^{|S|}$  possible outcomes.

Every subset of  $S$  can be constructed this way and different choices lead to different sets. Thus,  $|\mathcal{P}(S)| = 2^{|S|}$ . □



# Questions



Questions?

# Summary

# Summary

- Sets are unordered collections of distinct objects.
- Important set relations: equality ( $=$ ), subset ( $\subseteq$ ), superset ( $\supseteq$ ) and strict variants ( $\subset$  and  $\supset$ )
- The power set of a set  $S$  is the set of all subsets of  $S$ .
- Important set operations are intersection, union, set difference and complement.
  - Union and intersection are commutative and associative.
  - Union distributes over intersection and vice versa.
  - De Morgan's law for complement of union or intersection.
- The number of elements in a finite set is called its cardinality.

# Discrete Mathematics in Computer Science

## A3. Proofs: Introduction

Malte Helmert, Gabriele Röger

University of Basel

September 22, 2025

What is a Proof?

# What is a Proof?

A **mathematical proof** is

- a sequence of logical steps
- starting with one set of statements
- that comes to the conclusion  
that some statement must be true.

# What is a Proof?

A **mathematical proof** is

- a sequence of logical steps
- starting with one set of statements
- that comes to the conclusion  
that some statement must be true.

What is a **statement**?

# Mathematical Statements

## Mathematical Statement

A **mathematical statement** is a declarative sentence that is either true or false (but not both).

Examples (some true, some false):

- Let  $p \in \mathbb{N}_0$  be a prime number. Then  $p$  is odd.
- There exists an even prime number.
- The equation  $a^k + b^k = c^k$  has infinitely many solutions with  $a, b, c, k \in \mathbb{N}_1$  and  $k \geq 2$ .

German: Mathematische Aussage



# Mathematical Statements: Quantification

Statements often use **quantification**.

- Universal quantification:

“For all  $x$  in set  $S$  it holds that  $\langle \text{sub-statement on } x \rangle$ .”

This is **true** if the sub-statement is true for every  $x$  in  $S$ .

- Existential quantification:

“There is an  $x$  in set  $S$  such that  $\langle \text{sub-statement on } x \rangle$ .”

This is **true** if there exists at least one  $x$  in  $S$  for which the sub-statement is true.

Examples (some true, some false):

- For all  $x \in \mathbb{N}_1$  it holds that  $x + 1$  is in  $\mathbb{N}_1$ .
- For all  $x \in \mathbb{N}_1$  it holds that  $x - 1$  is in  $\mathbb{N}_1$ .
- There is an  $x \in \mathbb{N}_1$  such that  $x = \sqrt{x}$ .

# Mathematical Statements: Preconditions and Conclusions

We can identify **preconditions** and **conclusions**.

“If  $\langle$ preconditions $\rangle$  then  $\langle$ conclusions $\rangle$ .”

The statement is **true** if the conclusions are true whenever the preconditions are true.

Not every statement has preconditions. Preconditions are often used in universally quantified sub-statements.

Examples (some true, some false):

- If 4 is a prime number then  $2 \cdot 3 = 4$ .
- If  $n$  is a prime number with  $n > 2$  then  $n$  is odd.
- For all  $p \in \mathbb{N}_1$  it holds that if  $p$  is a prime number then  $p$  is odd.

## Different Statements with the same Meaning

The following statements have the same meaning, we just move preconditions into the quantification, make some aspects implicit, and change the structure.

- For all  $p \in \mathbb{N}_1$  it holds that if  $p$  is a prime number with  $p > 2$  then  $p$  is odd.
- For all prime numbers  $p$  it holds that if  $p > 2$  then  $p$  is odd.
- Let  $p$  be a natural number with  $p > 2$ .  
Then  $p$  is prime if  $p$  is odd.
- If  $p$  is a prime number with  $p > 2$  then  $p$  is odd.
- All prime numbers  $p > 2$  are odd.

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The following statements have the same meaning, we just move preconditions into the quantification, make some aspects implicit, and change the structure.

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- For all prime numbers  $p$  it holds that if  $p > 2$  then  $p$  is odd.
- Let  $p$  be a natural number with  $p > 2$ .  
Then  $p$  is prime if  $p$  is odd.
- If  $p$  is a prime number with  $p > 2$  then  $p$  is odd.
- All prime numbers  $p > 2$  are odd.

A single mathematical statement can be expressed in different ways, as long as the meaning stays the same.

Like paraphrasing a sentence in everyday language.

# On what Statements can we Build the Proof?

A mathematical proof is

- a sequence of logical steps
- **starting with one set of statements**
- that comes to the conclusion  
that some statement must be true.

We can use:

- **axioms**: statements that are assumed to always be true  
in the current context
- **theorems** and **lemmas**: statements that were already proven
  - lemma: an intermediate tool
  - theorem: itself a relevant result
- **premises**: assumptions we make  
to see what consequences they have

German: Axiom, Theorem/Satz, Lemma, Prämisse/Annahme

# What is a Logical Step?

A mathematical proof is

- a sequence of logical steps
- starting with one set of statements
- that comes to the conclusion  
that some statement must be true.

Each step directly follows

- from the axioms,
- premises,
- previously proven statements and
- the preconditions of the statement we want to prove.

# What is a Logical Step?

A mathematical proof is

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- starting with one set of statements
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Each step directly follows

- from the axioms,
- premises,
- previously proven statements and
- the preconditions of the statement we want to prove.

For a formal definition, we would need formal logics.

# The Role of Definitions

## Definition

A **set** is an unordered collection of distinct objects.

The objects in a set are called the **elements** of the set. A set is said to **contain** its elements.

We write  $x \in S$  to indicate that  $x$  is an element of set  $S$ , and  $x \notin S$  to indicate that  $S$  does not contain  $x$ .

The set that does not contain any objects is the **empty set**  $\emptyset$ .



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The set that does not contain any objects is the **empty set**  $\emptyset$ .

- A definition introduces an abbreviation.
- Whenever we say “set”, we could instead say “an unordered collection of distinct objects” and vice versa.
- Definitions can also introduce notation.

German: Definition

# Disproofs

- A **disproof** (**refutation**) shows that a given mathematical statement is **false** by giving an example where the preconditions are true, but the conclusion is false.
- This requires deriving, in a sequence of proof steps, the opposite (negation) of the conclusion.

German: Widerlegung

# Disproofs

- A **disproof** (**refutation**) shows that a given mathematical statement is **false** by giving an example where the preconditions are true, but the conclusion is false.
- This requires deriving, in a sequence of proof steps, the opposite (negation) of the conclusion.

## Example (False statement)

"If  $p \in \mathbb{N}_0$  is a prime number then  $p$  is odd."

## Refutation.

Consider natural number 2 as a counter example. It is prime because it has exactly 2 divisors, 1 and itself. It is not odd, because it is divisible by 2.



German: Widerlegung

## A Word on Style

A proof should help the reader to see why the result must be true.

- A proof should be easy to follow.
- Omit unnecessary information.
- Move self-contained parts into separate lemmas.
- In complicated proofs, reveal the overall structure in advance.
- Have a clear line of argument.

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→ Writing a proof is like writing an essay.

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A proof should help the reader to see why the result must be true.

- A proof should be easy to follow.
- Omit unnecessary information.
- Move self-contained parts into separate lemmas.
- In complicated proofs, reveal the overall structure in advance.
- Have a clear line of argument.

→ Writing a proof is like writing an essay.

Recommended reading (ADAM additional resources):

- “Some Remarks on Writing Mathematical Proofs” (John M. Lee)
- “§1. Minicourse on technical writing” of “Mathematical Writing” (Donald E. Knuth, Tracy Larrabee, and Paul M. Roberts)

# Questions



Questions?

# Summary



# Summary

A proof should convince the reader by **logical steps** of the truth of some mathematical statement.

# Discrete Mathematics in Computer Science

## A4. Proof Techniques I

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September 24, 2025

# Proof Strategies

# Common Forms of Statements

Many statements have one of these forms:

- ① “All  $x \in S$  with the property  $P$  also have the property  $Q$ .”
- ② “ $A$  is a subset of  $B$ .”
- ③ “For all  $x \in S$ :  $x$  has property  $P$  iff  $x$  has property  $Q$ .”  
(“iff”: “if and only if”)
- ④ “ $A = B$ ”, where  $A$  and  $B$  are sets.

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(“iff”: “if and only if”)
- ④ “ $A = B$ ”, where  $A$  and  $B$  are sets.

In the following, we will discuss some typical proof/disproof strategies for such statements.

# Proof Strategies

- 1 “All  $x \in S$  with the property  $P$  also have the property  $Q$ .”  
“For all  $x \in S$ : if  $x$  has property  $P$ , then  $x$  has property  $Q$ .”
  - To prove, assume you are given an arbitrary  $x \in S$  that has the property  $P$ .  
Give a sequence of proof steps showing that  $x$  must have the property  $Q$ .
  - To disprove, find a **counterexample**, i. e., find an  $x \in S$  that has property  $P$  but not  $Q$  and prove this.

# Proof Strategies

- ② “ $A$  is a subset of  $B$ .”
  - To prove, assume you have an arbitrary element  $x \in A$  and prove that  $x \in B$ .
  - To disprove, find an element in  $x \in A \setminus B$  and prove that  $x \in A \setminus B$ .

# Proof Strategies

- ③ “For all  $x \in S$ :  $x$  has property  $P$  iff  $x$  has property  $Q$ .”  
(“iff”: “if and only if”)
  - To prove, separately prove “if  $P$  then  $Q$ ” and “if  $Q$  then  $P$ ”.
  - To disprove, disprove “if  $P$  then  $Q$ ” or disprove “if  $Q$  then  $P$ ”.



# Proof Strategies

- ④ “ $A = B$ ”, where  $A$  and  $B$  are sets.
  - To prove, separately prove “ $A \subseteq B$ ” and “ $B \subseteq A$ ”.
  - To disprove, disprove “ $A \subseteq B$ ” or disprove “ $B \subseteq A$ ”.

# Proof Techniques

most common proof techniques:

- direct proof
- indirect proof (proof by contradiction)
- contrapositive
- mathematical induction
- structural induction

# Direct Proof

# Direct Proof

## Direct Proof

Direct derivation of the statement by deducing or rewriting.

German: Direkter Beweis

## Direct Proof: Example

### Theorem

*For all sets  $A$ ,  $B$  and  $C$  it holds that*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

### Proof.

## Direct Proof: Example

### Theorem

*For all sets  $A$ ,  $B$  and  $C$  it holds that*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

### Proof.

Let  $A$ ,  $B$  and  $C$  be arbitrary sets.

## Direct Proof: Example

### Theorem

*For all sets  $A$ ,  $B$  and  $C$  it holds that*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

### Proof.

Let  $A$ ,  $B$  and  $C$  be arbitrary sets.

We will show separately that

- $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$  and that
- $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ .

...

## Direct Proof: Example cont.

Proof (continued).

We first show that  $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ :



## Direct Proof: Example cont.

Proof (continued).

We first show that  $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ :

If  $A \cap (B \cup C)$  is empty, the statement is trivially true. Otherwise consider an arbitrary  $x \in A \cap (B \cup C)$ .

## Direct Proof: Example cont.

Proof (continued).

We first show that  $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ :

If  $A \cap (B \cup C)$  is empty, the statement is trivially true. Otherwise consider an arbitrary  $x \in A \cap (B \cup C)$ . By the definition of the intersection it holds that  $x \in A$  and that  $x \in (B \cup C)$ .

## Direct Proof: Example cont.

Proof (continued).

We first show that  $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ :

If  $A \cap (B \cup C)$  is empty, the statement is trivially true. Otherwise consider an arbitrary  $x \in A \cap (B \cup C)$ . By the definition of the intersection it holds that  $x \in A$  and that  $x \in (B \cup C)$ .

We make a case distinction between  $x \in B$  and  $x \notin B$ :

## Direct Proof: Example cont.

Proof (continued).

We first show that  $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ :

If  $A \cap (B \cup C)$  is empty, the statement is trivially true. Otherwise consider an arbitrary  $x \in A \cap (B \cup C)$ . By the definition of the intersection it holds that  $x \in A$  and that  $x \in (B \cup C)$ .

We make a case distinction between  $x \in B$  and  $x \notin B$ :

**Case 1 ( $x \in B$ ):** As  $x \in A$  is true, it holds in this case that  $x \in (A \cap B)$ .

## Direct Proof: Example cont.

Proof (continued).

We first show that  $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ :

If  $A \cap (B \cup C)$  is empty, the statement is trivially true. Otherwise consider an arbitrary  $x \in A \cap (B \cup C)$ . By the definition of the intersection it holds that  $x \in A$  and that  $x \in (B \cup C)$ .

We make a case distinction between  $x \in B$  and  $x \notin B$ :

**Case 1 ( $x \in B$ ):** As  $x \in A$  is true, it holds in this case that  $x \in (A \cap B)$ .

**Case 2 ( $x \notin B$ ):** From  $x \in (B \cup C)$  it follows for this case that  $x \in C$ . With  $x \in A$  we conclude that  $x \in (A \cap C)$ .

## Direct Proof: Example cont.

Proof (continued).

We first show that  $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ :

If  $A \cap (B \cup C)$  is empty, the statement is trivially true. Otherwise consider an arbitrary  $x \in A \cap (B \cup C)$ . By the definition of the intersection it holds that  $x \in A$  and that  $x \in (B \cup C)$ .

We make a case distinction between  $x \in B$  and  $x \notin B$ :

**Case 1 ( $x \in B$ ):** As  $x \in A$  is true, it holds in this case that  $x \in (A \cap B)$ .

**Case 2 ( $x \notin B$ ):** From  $x \in (B \cup C)$  it follows for this case that  $x \in C$ . With  $x \in A$  we conclude that  $x \in (A \cap C)$ .

In both cases it holds that  $x \in A \cap B$  or  $x \in A \cap C$ , and we conclude that  $x \in (A \cap B) \cup (A \cap C)$ .

## Direct Proof: Example cont.

Proof (continued).

We first show that  $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ :

If  $A \cap (B \cup C)$  is empty, the statement is trivially true. Otherwise consider an arbitrary  $x \in A \cap (B \cup C)$ . By the definition of the intersection it holds that  $x \in A$  and that  $x \in (B \cup C)$ .

We make a case distinction between  $x \in B$  and  $x \notin B$ :

**Case 1 ( $x \in B$ ):** As  $x \in A$  is true, it holds in this case that  $x \in (A \cap B)$ .

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In both cases it holds that  $x \in A \cap B$  or  $x \in A \cap C$ , and we conclude that  $x \in (A \cap B) \cup (A \cap C)$ .

As  $x$  was chosen arbitrarily from  $A \cap (B \cup C)$ , we have shown that every element of  $A \cap (B \cup C)$  is an element of  $(A \cap B) \cup (A \cap C)$ , so it holds that  $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ . ...

## Direct Proof: Example cont.

Proof (continued).

We will now show that  $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ .

... **[Homework assignment]** ...

Overall we have shown for arbitrary sets  $A, B$  and  $C$  that  $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$  and that  $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ , which concludes the proof of the theorem. □



# Indirect Proof

# Indirect Proof

## Indirect Proof (Proof by Contradiction)

- Make an **assumption** that the statement is false.
- Use the assumption to derive a **contradiction**.
- This shows that the assumption must be false and hence the original statement must be true.

German: Indirekter Beweis, Beweis durch Widerspruch

## Indirect Proof: Example

### Theorem

*Let  $A$  and  $B$  be sets. If  $A \setminus B = \emptyset$  then  $A \subseteq B$ .*

## Indirect Proof: Example

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*Let  $A$  and  $B$  be sets. If  $A \setminus B = \emptyset$  then  $A \subseteq B$ .*

### Proof.

We prove the theorem by contradiction.

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### Theorem

*Let  $A$  and  $B$  be sets. If  $A \setminus B = \emptyset$  then  $A \subseteq B$ .*

### Proof.

We prove the theorem by contradiction.

Assume that there are sets  $A$  and  $B$  with  $A \setminus B = \emptyset$  and  $A \not\subseteq B$ .

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We prove the theorem by contradiction.

Assume that there are sets  $A$  and  $B$  with  $A \setminus B = \emptyset$  and  $A \not\subseteq B$ .

Let  $A$  and  $B$  be such sets.

## Indirect Proof: Example

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We prove the theorem by contradiction.

Assume that there are sets  $A$  and  $B$  with  $A \setminus B = \emptyset$  and  $A \not\subseteq B$ .

Let  $A$  and  $B$  be such sets.

Since  $A \not\subseteq B$  there is some  $x \in A$  such that  $x \notin B$ .

## Indirect Proof: Example

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For this  $x$  it holds that  $x \in A \setminus B$ .



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We prove the theorem by contradiction.

Assume that there are sets  $A$  and  $B$  with  $A \setminus B = \emptyset$  and  $A \not\subseteq B$ .

Let  $A$  and  $B$  be such sets.

Since  $A \not\subseteq B$  there is some  $x \in A$  such that  $x \notin B$ .

For this  $x$  it holds that  $x \in A \setminus B$ .

This is a contradiction to  $A \setminus B = \emptyset$ .

We conclude that the assumption was false and thus the theorem is true. □

# Proof by Contrapositive

# Contrapositive

(Proof by) Contrapositive

Prove “If  $A$ , then  $B$ ” by proving “If not  $B$ , then not  $A$ .”

# Contrapositive

## (Proof by) Contrapositive

Prove “If  $A$ , then  $B$ ” by proving “If not  $B$ , then not  $A$ .”

### Examples:

- Prove “For all  $n \in \mathbb{N}_0$ : if  $n^2$  is odd, then  $n$  is odd” by proving “For all  $n \in \mathbb{N}_0$ , if  $n$  is even, then  $n^2$  is even.”
- Prove “For all  $n \in \mathbb{N}_0$ : if  $n$  is not a square number, then  $\sqrt{n}$  is irrational” by proving “For all  $n \in \mathbb{N}_0$ : if  $\sqrt{n}$  is rational, then  $n$  is a square number.”

German: Kontraposition

## Contrapositive: Example

### Theorem

*For any sets  $A$  and  $B$ : If  $A \subseteq B$  then  $A \setminus B = \emptyset$ .*

## Contrapositive: Example

### Theorem

*For any sets  $A$  and  $B$ : If  $A \subseteq B$  then  $A \setminus B = \emptyset$ .*

### Proof.

We prove the theorem by contrapositive, showing for any sets  $A$  and  $B$  that if  $A \setminus B \neq \emptyset$  then  $A \not\subseteq B$ .

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### Theorem

*For any sets  $A$  and  $B$ : If  $A \subseteq B$  then  $A \setminus B = \emptyset$ .*

### Proof.

We prove the theorem by contrapositive, showing for any sets  $A$  and  $B$  that if  $A \setminus B \neq \emptyset$  then  $A \not\subseteq B$ .

Let  $A$  and  $B$  be arbitrary sets with  $A \setminus B \neq \emptyset$ .

## Contrapositive: Example

### Theorem

*For any sets  $A$  and  $B$ : If  $A \subseteq B$  then  $A \setminus B = \emptyset$ .*

### Proof.

We prove the theorem by contrapositive, showing for any sets  $A$  and  $B$  that if  $A \setminus B \neq \emptyset$  then  $A \not\subseteq B$ .

Let  $A$  and  $B$  be arbitrary sets with  $A \setminus B \neq \emptyset$ .

As the set difference is not empty, there is at least one  $x$  with  $x \in A \setminus B$ .



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Let  $A$  and  $B$  be arbitrary sets with  $A \setminus B \neq \emptyset$ .

As the set difference is not empty, there is at least one  $x$  with  $x \in A \setminus B$ . By the definition of the set difference ( $\setminus$ ), it holds for such  $x$  that  $x \in A$  and  $x \notin B$ .

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Let  $A$  and  $B$  be arbitrary sets with  $A \setminus B \neq \emptyset$ .

As the set difference is not empty, there is at least one  $x$  with  $x \in A \setminus B$ . By the definition of the set difference ( $\setminus$ ), it holds for such  $x$  that  $x \in A$  and  $x \notin B$ .

Hence, not all elements of  $A$  are elements of  $B$ , so it does not hold that  $A \subseteq B$ . □

# Questions



Questions?

# Summary

# Summary

- There are standard strategies for proving some common forms of statements, e.g. some property of all elements of a set.
- **Direct proof**: derive statement by deducing or rewriting.
- **Indirect proof**: derive contradiction from the assumption that the statement is false.
- **Proof by contrapositive**: Prove “If A, then B” by proving “If not B, then not A.”.

# Discrete Mathematics in Computer Science

## A5. Proof Techniques II

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September 29, 2025

# Mathematical Induction

# Proof Techniques

most common proof techniques:

- direct proof
- indirect proof (proof by contradiction)
- contrapositive
- mathematical induction
- structural induction



# Mathematical Induction

Concrete Mathematics by Graham, Knuth and Patashnik (p. 3)

Mathematical induction proves that

we can climb as high as we like on a ladder,

by proving that we can climb onto the bottom rung (the basis)

and that

from each rung we can climb up to the next one (the step).

# Propositions

Consider a statement on all natural numbers  $n$  with  $n \geq m$ .

- E.g. “Every natural number  $n \geq 2$  can be written as a product of prime numbers.”
  - $P(2)$ : “2 can be written as a product of prime numbers.”
  - $P(3)$ : “3 can be written as a product of prime numbers.”
  - $P(4)$ : “4 can be written as a product of prime numbers.”
  - ...
  - $P(n)$ : “ $n$  can be written as a product of prime numbers.”
  - For every natural number  $n \geq 2$  proposition  $P(n)$  is true.

**Proposition**  $P(n)$  is a mathematical statement that is defined in terms of natural number  $n$ .

# Mathematical Induction

## Mathematical Induction

Proof (of the truth) of proposition  $P(n)$   
for all natural numbers  $n$  with  $n \geq m$ :

- **basis**: proof of  $P(m)$
- **induction hypothesis** (IH):  
suppose that  $P(k)$  is true for all  $k$  with  $m \leq k \leq n$
- **inductive step**: proof of  $P(n+1)$   
using the induction hypothesis

German: Vollständige Induktion, Induktionsanfang,  
Induktionsannahme oder Induktionsvoraussetzung,  
Induktionsschritt

# Mathematical Induction: Example I

## Theorem

*Every natural number  $n \geq 2$  can be written as a product of prime numbers, i. e.  $n = p_1 \cdot p_2 \cdot \dots \cdot p_m$  with prime numbers  $p_1, \dots, p_m$ .*

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## Proof.

Mathematical Induction over  $n$ :

basis  $n = 2$ : trivially satisfied, since 2 is prime

...

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## Proof (continued).

inductive step  $n \rightarrow n + 1$ :

- Case 1:  $n + 1$  is a prime number  $\rightsquigarrow$  trivial



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## Proof (continued).

inductive step  $n \rightarrow n + 1$ :

- **Case 1:**  $n + 1$  is a prime number  $\rightsquigarrow$  trivial
- **Case 2:**  $n + 1$  is not a prime number.

There are natural numbers  $2 \leq q, r \leq n$  with  $n + 1 = q \cdot r$ .

Using the IH shows that there are prime numbers

$q_1, \dots, q_s$  with  $q = q_1 \cdot \dots \cdot q_s$  and

$r_1, \dots, r_t$  with  $r = r_1 \cdot \dots \cdot r_t$ .

Together this means  $n + 1 = q_1 \cdot \dots \cdot q_s \cdot r_1 \cdot \dots \cdot r_t$ .





## Mathematical Induction: Example II

### Theorem

*Let  $S$  be a finite set. Then  $|\mathcal{P}(S)| = 2^{|S|}$ .*

What proposition can we use to prove this with mathematical induction?

## Proof by Induction

Proof.

By induction over  $|S|$ .

**Basis ( $|S| = 0$ ):** Then  $S = \emptyset$  and  $|\mathcal{P}(S)| = |\{\emptyset\}| = 1 = 2^0$ .

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**Inductive Step ( $n \rightarrow n + 1$ ):**

Let  $S'$  be an arbitrary set with  $|S'| = n + 1$  and let  $e$  be an arbitrary member of  $S'$ .

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Let  $S'$  be an arbitrary set with  $|S'| = n + 1$  and let  $e$  be an arbitrary member of  $S'$ .

Let further  $S = S' \setminus \{e\}$  and  $X = \{S'' \cup \{e\} \mid S'' \in \mathcal{P}(S)\}$ .

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Then  $\mathcal{P}(S') = \mathcal{P}(S) \cup X$ . As  $\mathcal{P}(S)$  and  $X$  are disjoint and  $|X| = |\mathcal{P}(S)|$ , it holds that  $|\mathcal{P}(S')| = 2|\mathcal{P}(S)|$ .

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Then  $\mathcal{P}(S') = \mathcal{P}(S) \cup X$ . As  $\mathcal{P}(S)$  and  $X$  are disjoint and  $|X| = |\mathcal{P}(S)|$ , it holds that  $|\mathcal{P}(S')| = 2|\mathcal{P}(S)|$ .

Since  $|S| = n$ , we can use the IH and get

$$|\mathcal{P}(S')| = 2 \cdot 2^{|S|} = 2 \cdot 2^n = 2^{n+1} = 2^{|S'|}.$$



# Weak vs. Strong Induction

- **Weak induction:** Induction hypothesis only supposes that  $P(k)$  is true for  $k = n$
- **Strong induction:** Induction hypothesis supposes that  $P(k)$  is true for all  $k \in \mathbb{N}_0$  with  $m \leq k \leq n$ 
  - also: **complete induction**



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- **Weak induction:** Induction hypothesis only supposes that  $P(k)$  is true for  $k = n$
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  - also: **complete induction**

Our previous definition corresponds to **strong induction**.

Which of the examples had also worked with weak induction?

# Is Strong Induction More Powerful than Weak Induction?

Are there statements that we can prove with strong induction but not with weak induction?

# Is Strong Induction More Powerful than Weak Induction?

Are there statements that we can prove with strong induction but not with weak induction?

We can always use a stronger proposition:

- “Every  $n \in \mathbb{N}_0$  with  $n \geq 2$  can be written as a product of prime numbers.”
- $P(n)$ : “ $n$  can be written as a product of prime numbers.”
- $P'(n)$ : “all  $k \in \mathbb{N}_0$  with  $2 \leq k \leq n$  can be written as a product of prime numbers.”

# Questions



Questions?

# Structural Induction

# Inductively Defined Sets: Examples

## Example (Natural Numbers)

The set  $\mathbb{N}_0$  of natural numbers is inductively defined as follows:

- 0 is a natural number.
- If  $n$  is a natural number, then  $n + 1$  is a natural number.

German: Binärbaum, Blatt, innerer Knoten

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## Example (Binary Tree)

The set  $\mathcal{B}$  of binary trees is inductively defined as follows:

- $\square$  is a binary tree (a leaf)
- If  $L$  and  $R$  are binary trees, then  $\langle L, \bigcirc, R \rangle$  is a binary tree (with inner node  $\bigcirc$ ).

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**Implicit statement:** all elements of the set can be constructed  
by finite application of these rules

German: Binärbaum, Blatt, innerer Knoten

# Inductive Definition of a Set

## Inductive Definition

A set  $M$  can be defined **inductively** by specifying

- **basic elements** that are contained in  $M$
- **construction rules** of the form  
“Given some elements of  $M$ , another element of  $M$  can be constructed like this.”

German: Induktive Definition, Basiselemente, Konstruktionsregeln

# Structural Induction

## Structural Induction

Proof of statement for all elements of an inductively defined set

- **basis**: proof of the statement for the basic elements
- **induction hypothesis (IH)**:  
suppose that the statement is true for some elements  $M$
- **inductive step**: proof of the statement for elements constructed by applying a construction rule to  $M$   
(one inductive step for each construction rule)

German: Strukturelle Induktion

## Structural Induction: Example (1)

### Definition (Leaves of a Binary Tree)

The number of **leaves** of a binary tree  $B$ , written  $leaves(B)$ , is defined as follows:

$$leaves(\square) = 1$$

$$leaves(\langle L, \bigcirc, R \rangle) = leaves(L) + leaves(R)$$

### Definition (Inner Nodes of a Binary Tree)

The number of **inner nodes** of a binary tree  $B$ , written  $inner(B)$ , is defined as follows:

$$inner(\square) = 0$$

$$inner(\langle L, \bigcirc, R \rangle) = inner(L) + inner(R) + 1$$

## Structural Induction: Example (2)

### Theorem

*For all binary trees  $B$ :  $\text{inner}(B) = \text{leaves}(B) - 1$ .*

## Structural Induction: Example (2)

### Theorem

*For all binary trees  $B$ :  $inner(B) = leaves(B) - 1$ .*

### Proof.

induction basis:

$$inner(\square) = 0 = 1 - 1 = leaves(\square) - 1$$

$\leadsto$  statement is true for base case

...

## Structural Induction: Example (3)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree  $\langle L, \bigcirc, R \rangle$ ,  
we may use that it is true for the subtrees  $L$  and  $R$ .



## Structural Induction: Example (3)

Proof (continued).

induction hypothesis:

to prove that the statement is true for a composite tree  $\langle L, \bigcirc, R \rangle$ , we may use that it is true for the subtrees  $L$  and  $R$ .

inductive step for  $B = \langle L, \bigcirc, R \rangle$ :

$$\begin{aligned} inner(B) &= inner(L) + inner(R) + 1 \\ &\stackrel{\text{IH}}{=} (leaves(L) - 1) + (leaves(R) - 1) + 1 \\ &= leaves(L) + leaves(R) - 1 = leaves(B) - 1 \end{aligned}$$





## Example: Tarradiddles

### Example (Tarradiddles)

The set of tarradiddles is inductively defined as follows:

- ✈ is a tarradiddle.
- ♥ is a tarradiddle.
- If  $x$  and  $y$  are tarradiddles, then  $x\text{✿✿}y$  is a tarradiddle.
- If  $x$  and  $y$  are tarradiddles, then  $\text{✿}x\text{✈}y\text{✿}$  is a tarradiddle.

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- If  $x$  and  $y$  are tarradiddles, then  $\text{🌸}x\text{✈}y\text{🌸}$  is a tarradiddle.

How do you prove with structural induction that every tarradiddle contains an even number of flowers?

# Questions



Questions?

# Excursus: Computer-assisted Theorem Proving

# Computer-assisted Proofs

- Computers can help proving theorems.
- **Computer-aided proofs** have for example been used for proving theorems by exhaustion.
- Example: **Four color theorem**

# Interactive Theorem Proving

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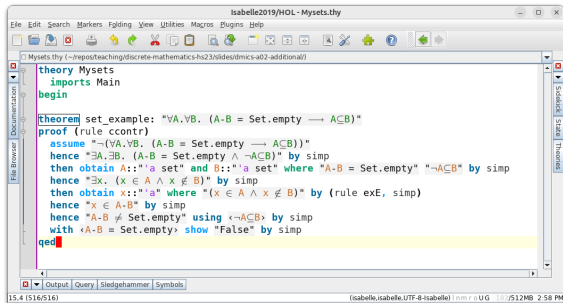
# Interactive Theorem Proving

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- Example theorem provers: Isabelle/HOL, Lean

# Example



The screenshot shows the Isabelle2019/HOL - Mysets.thy editor. The main window displays the following code:

```
theory Mysets
  imports Main
begin

theorem set_example: "∀A.∀B. (A-B = Set.empty ⟶ A⊆B)"
proof (rule ccontr)
  assume "¬(∀A.∀B. (A-B = Set.empty ⟶ A⊆B))"
  hence "∃A.∃B. (A-B = Set.empty ∧ ¬A⊆B)" by simp
  then obtain A::"a set" and B::"a set" where "A-B = Set.empty" "¬A⊆B" by simp
  hence "∃x. (x ∈ A ∧ x ∉ B)" by simp
  then obtain x::"a" where "(x ∈ A ∧ x ∉ B)" by (rule exE, simp)
  hence "x ∈ A-B" by simp
  hence "A-B ≠ Set.empty" using "x ∈ A-B" by simp
  with "A-B = Set.empty" show "False" by simp
qed
```

The status bar at the bottom indicates the version is 15.4 (516/516) and the session is (isabelle.isabelle.UTF-8-isabelle) in m r o U G, with 107/512MB of memory used at 2:58 PM.

→ Demo

# Summary

# Summary

- **Mathematical induction** is used to prove a proposition  $P$  for all natural numbers  $\geq m$ .
  - Prove  $P(m)$ .
  - Make hypothesis that  $P(k)$  is true for  $m \leq k \leq n$ .
  - Establish  $P(n+1)$  using the hypothesis.
- **Structural induction** applies the same general concept to prove a proposition  $P$  for all elements of an inductively defined set.

# Discrete Mathematics in Computer Science

## B1. Tuples & Cartesian Product

Malte Helmert, Gabriele Röger

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October 1, 2025

# Tuples and the Cartesian Product



# Motivation

- A **set** is an **unordered collection** of **distinct** objects.
- We often need a more structured way of representation.
  - A person is associated with a name, address, phone number.
  - A set of persons makes sense in many contexts.
  - Representing the associated data as a set rather not.
- We could for example want to
  - directly access the name of a person, or
  - have a separate billing and delivery address for some order, but in general, these can be the same.
- **Tuples** are mathematical building blocks that support this.

# Sets vs. Tuples

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# Sets vs. Tuples

- A **set** is an **unordered collection** of **distinct** objects.
- A **tuple** is an **ordered sequence** of objects.

# Tuples

- **$k$ -tuple**: ordered sequence of  $k$  objects ( $k \in \mathbb{N}_0$ )
- written  $(o_1, \dots, o_k)$  or  $\langle o_1, \dots, o_k \rangle$
- unlike sets, **order matters** ( $\langle 1, 2 \rangle \neq \langle 2, 1 \rangle$ )
- objects may occur multiple times in a tuple

German:  $k$ -Tupel, Komponente, (geordnetes) Paar, Tripel, Quadrupel

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- unlike sets, **order matters** ( $\langle 1, 2 \rangle \neq \langle 2, 1 \rangle$ )
- objects may occur multiple times in a tuple
- objects contained in tuples are called **components**
- terminology:
  - $k = 2$ : (ordered) pair
  - $k = 3$ : triple
  - more rarely: quadruple, quintuple, sextuple, septuple, ...
- if  $k$  is clear from context (or does not matter), often just called **tuple**

German:  $k$ -Tupel, Komponente, (geordnetes) Paar, Tripel, Quadrupel

# Equality of Tuples

## Definition (Equality of Tuples)

Two  $n$ -tuples  $t = \langle o_1, \dots, o_n \rangle$  and  $t' = \langle o'_1, \dots, o'_n \rangle$  are **equal** ( $t = t'$ ) if for  $i \in \{1, \dots, n\}$  it holds that  $o_i = o'_i$ .

# Cartesian Product

## Definition (Cartesian Product and Cartesian Power)

Let  $S_1, \dots, S_n$  be sets. The **Cartesian product**  $S_1 \times \dots \times S_n$  is the following set of  $n$ -tuples:

$$S_1 \times \dots \times S_n = \{ \langle x_1, \dots, x_n \rangle \mid x_1 \in S_1, x_2 \in S_2, \dots, x_n \in S_n \}.$$

**René Descartes:** French mathematician and philosopher (1596–1650)

German: Kartesisches Produkt

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**Example:**  $A = \{a, b\}$ ,  $B = \{1, 2, 3\}$

$A \times B =$

German: Kartesisches Produkt



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The  $k$ -ary **Cartesian power** of a set  $S$  (with  $k \in \mathbb{N}_1$ ) is the set  $S^k = \{ \langle o_1, \dots, o_k \rangle \mid o_i \in S \text{ for all } i \in \{1, \dots, k\} \} = \underbrace{S \times \dots \times S}_{k \text{ times}}.$

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**Example:**  $A = \{a, b\}$ ,  $B = \{1, 2, 3\}$

$$A^2 =$$

German: Kartesisches Produkt

# (Non-)properties of the Cartesian Product

The Cartesian product is

- **not commutative**, in most cases  $A \times B \neq B \times A$ .
- **not associative**, in most cases  $(A \times B) \times C \neq A \times (B \times C)$

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Why? Exceptions?

# Questions



Questions?

# Summary

# Summary

- A  $k$ -tuple is an ordered sequence of  $k$  objects, called the components of the tuple.
- 2-tuples are also called pairs and 3-tuples triples.
- The Cartesian Product  $S_1 \times \cdots \times S_n$  of set  $S_1, \dots, S_n$  is the set of all tuples  $\langle o_1, \dots, o_n \rangle$ , where for all  $i \in \{1, \dots, n\}$  component  $o_i$  is an element of  $S_i$ .

# Discrete Mathematics in Computer Science

## B2. Relations

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October 6, 2025

# Relations



## Relations: Informally

- Intuitively, a mathematical relation connects elements from several (possibly different) sets by specifying related groupings.
- We already know some relations, e. g.
  - $\subseteq$  relation for sets
  - $\leq$  relation for natural numbers

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- These are examples of **binary** relations, considering **pairs of objects**.
- There are also relations of **higher arity**, e. g.
  - " $x + y = z$ " for integers  $x, y, z$ .
  - "The name, address and office number belong to the same person."

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- There are also relations of **higher arity**, e. g.
  - " $x + y = z$ " for integers  $x, y, z$ .
  - "The name, address and office number belong to the same person."
- Relations are for example important for relational databases, semantic networks or knowledge representation and reasoning.

# Relations

## Definition (Relation)

Let  $S_1, \dots, S_n$  be sets.

A **relation over  $S_1, \dots, S_n$**  is a set  $R \subseteq S_1 \times \dots \times S_n$ .

The **arity** of  $R$  is  $n$ .

A relation of arity  $n$  is a set of  $n$ -tuples.

German: Relation, Stelligkeit

## Relations: Examples

- $\subseteq = \{(S, S') \mid S \text{ and } S' \text{ are sets and for every } x \in S \text{ it holds that } x \in S'\}$

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## Relations: Examples

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- $R' = \{(\text{Gabi Röger, Spiegelgasse 1, 04.005}),$   
     $(\text{Malte Helmert, Spiegelgasse 1, 06.004}),$   
     $(\text{David Speck, Spiegelgasse 5, 04.003})\}$



# Questions



Questions?

# Properties of Binary Relations

# Binary Relation

A binary relation is a relation of arity 2:

Definition (binary relation)

A **binary relation** is a relation over two sets  $A$  and  $B$ .

German: zweistellige Relation, homogene Relation

# Binary Relation

A binary relation is a relation of arity 2:

## Definition (binary relation)

A **binary relation** is a relation over two sets  $A$  and  $B$ .

- Instead of  $(x, y) \in R$ , we also write  $xRy$ , e. g.  
 $x \leq y$  instead of  $(x, y) \in \leq$
- If the sets are equal, we say “ $R$  is a binary relation over  $A$ ” instead of “ $R$  is a binary relation over  $A$  and  $A$ ”.
- Such a relation over a set is also called a **homogeneous relation** or an **endorelation**.

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# Reflexivity

A **reflexive** relation relates every object to itself.

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# Asymmetry and Antisymmetry

How do these properties relate to irreflexivity?

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A binary relation  $R$  over set  $A$  is **transitive** if it holds for all  $a, b, c \in A$  that  
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- A binary relation over set  $S$  is a relation  $R \subseteq S \times S$  and also called a **homogeneous relation**.
- A binary relation  $R$  over  $A$  is
  - **reflexive** if  $(a, a) \in R$  for all  $a \in A$ ,
  - **irreflexive** if  $(a, a) \notin R$  for all  $a \in A$ ,
  - **symmetric** if for all  $a, b \in A$  it holds that  $(a, b) \in R$  iff  $(b, a) \in R$ ,
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# Special Classes of Relations

- Some important classes of relations are defined in terms of these properties.
  - **Equivalence relation:** reflexive, symmetric, transitive
  - **Partial order:** reflexive, antisymmetric, transitive
  - **Strict order:** irreflexive, asymmetric, transitive
  - ...
- We will consider these and other classes in detail.

# Discrete Mathematics in Computer Science

## B3. Equivalence and Order Relations

Malte Helmert, Gabriele Röger

University of Basel

October 6/8, 2025

# Equivalence Relations

# Motivation

- Think of any attribute that two objects can have in common, e. g. their color.
- We could place the objects into distinct “buckets”, e. g. one bucket for each color.
- We also can define a relation  $\sim$  such that  $x \sim y$  iff  $x$  and  $y$  share the attribute, e. g. have the same color.
- Would this relation be
  - reflexive?
  - irreflexive?
  - symmetric?
  - asymmetric?
  - antisymmetric?
  - transitive?

# Equivalence Relation

## Definition (Equivalence Relation)

A binary relation  $\sim$  over set  $S$  is an **equivalence relation** if  $\sim$  is **reflexive, symmetric and transitive**.

German: Äquivalenzrelation

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### Examples:

- $\{(x, y) \mid x \text{ and } y \text{ have the same place of origin}\}$   
over the set of all Swiss citizens
- $\{(x, y) \mid x \text{ and } y \text{ have the same parity}\}$  over  $\mathbb{N}_0$
- $\{(1, 1), (1, 4), (1, 5), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), (2, 2), (2, 3), (3, 2), (3, 3)\}$  over  $\{1, 2, \dots, 5\}$

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### Examples:

- $\{(x, y) \mid x \text{ and } y \text{ have the same place of origin}\}$   
over the set of all Swiss citizens
- $\{(x, y) \mid x \text{ and } y \text{ have the same parity}\}$  over  $\mathbb{N}_0$
- $\{(1, 1), (1, 4), (1, 5), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), (2, 2), (2, 3), (3, 2), (3, 3)\}$  over  $\{1, 2, \dots, 5\}$

Is this definition indeed what we want?

Does it allow us to partition the objects into buckets  
(e. g. one “bucket” for all objects that share a specific color)?

German: Äquivalenzrelation

# Equivalence Classes

## Definition (Equivalence Class)

Let  $\sim$  be an equivalence relation over set  $S$ .

For any  $x \in S$ , the **equivalence class of  $x$**  is the set

$$[x]_{\sim} = \{y \in S \mid x \sim y\}.$$

German: Äquivalenzklasse



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Consider

$$\sim = \{(1, 1), (1, 4), (1, 5), (4, 1), (4, 4), (4, 5), (5, 1), (5, 4), (5, 5), \\ (2, 2), (2, 3), (3, 2), (3, 3)\}$$

over set  $\{1, 2, \dots, 5\}$ .

$$[4]_{\sim} =$$

German: Äquivalenzklasse

# Equivalence Classes: Properties

Let  $\sim$  be an equivalence relation over set  $S$  and  $E = \{[x]_{\sim} \mid x \in S\}$  the set of all equivalence classes.

- Every element of  $S$  is in some equivalence class in  $E$ .
- Every element of  $S$  is in at most one equivalence class in  $E$ .  
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$\Rightarrow$  Equivalence relations induce partitions  
(not covered in this course).

# Questions



Questions?

# Order Relations

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- We now consider other combinations of properties, that allow us to describe a consistent order of the objects.

German: Ordnungsrelation

# Order Relations

- We now consider other combinations of properties, that allow us to describe a consistent order of the objects.
- “Number  $x$  is not larger than number  $y$ .”  
“Set  $S$  is a subset of set  $T$ .”  
“Jerry runs at least as fast as Tom.”  
“Pasta tastes better than Potatoes.”

German: Ordnungsrelation

# Partial Orders

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  - symmetric?
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Which of these relations are partial orders?

- strict subset relation  $\subset$  for sets
- not-less-than relation  $\geq$  over  $\mathbb{N}_0$
- $R = \{(a, a), (a, b), (b, b), (b, c), (c, c)\}$  over  $\{a, b, c\}$

German: Halbordnung oder partielle Ordnung

# Least and Greatest Element

## Definition (Least and greatest element)

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- Is there a least/greatest element? Which one?
  - $S = \{1, 2, 3\}$  and  $\preceq = \{(x, y) \mid x, y \in S \text{ and } x \leq y\}$

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  - relation  $\leq$  over  $\mathbb{Z}$
- Why can we say **the** least element instead of **a** least element?

German: kleinstes/grösstes Element

# Uniqueness of Least Element

## Theorem

*Let  $\preceq$  be a partial order over set  $S$ .*

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Analogously: If there is a greatest element then is unique.

# Minimal and Maximal Elements

## Definition (Minimal/Maximal element of a set)

Let  $\preceq$  be a partial order over set  $S$ .

An element  $x \in S$  is a **minimal element** of  $S$   
if **there is no  $y \in S$  with  $y \preceq x$  and  $x \neq y$ .**

An element  $x \in S$  is a **maximal element** of  $S$   
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German: minimales/maximales Element

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A set can have several minimal elements and no least element.

Example?

German: minimales/maximales Element



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  - $\{1, 2\} \not\subseteq \{2, 3\}$  and  $\{2, 3\} \not\subseteq \{1, 2\}$
- Relation  $\leq$  is a **total** order, relation  $\subseteq$  is not.

# Total Order – Definition

## Definition (Total relation)

A binary relation  $R$  over set  $S$  is **total** if for all  $x, y \in S$  at least one of  $xRy$  or  $yRx$  is true.

German: totale Relation

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## Definition (Total order)

A binary relation is a **total order** if it is **total** and a **partial order**.

German: totale Relation, (schwache) Totalordnung oder totale Ordnung

# Questions



Questions?



# Strict Orders

- A **partial** order is reflexive, antisymmetric and transitive.
- We now consider **strict orders**.

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## Strict Orders – Definition

### Definition (Strict (partial) order)

A binary relation  $\prec$  over set  $S$  is a **strict (partial) order** if  $\prec$  is **irreflexive, asymmetric and transitive**.

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Can a relation be both, a partial order and a strict (partial) order?

We can omit irreflexivity or asymmetry from the definition (but not both). Why?

German: strenge (Halb-)ordnung

## Strict Total Orders

- As partial orders, a strict order does not automatically allow us to rank arbitrary two objects against each other.

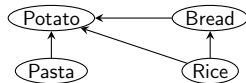


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- **Example 1** (personal preferences):

- "Pasta tastes better than potato."
- "Rice tastes better than bread."
- "Bread tastes better than potato."
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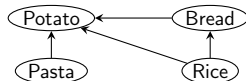
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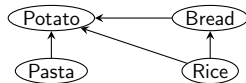
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  - No ranking of pasta against rice or of pasta against bread.
- **Example 2:**  $\subset$  relation for sets
  - It **doesn't work** to simply require that the strict order is total.  
Why?

## Strict Total Orders – Definition

### Definition (Trichotomy)

A binary relation  $R$  over set  $S$  is **trichotomous** if for all  $x, y \in S$  exactly one of  $xRy$ ,  $yRx$  or  $x = y$  is true.

German: trichotom

# Strict Total Orders – Definition

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## Definition (Strict total order)

A binary relation  $\prec$  over  $S$  is a **strict total order** if  $\prec$  is **trichotomous** and a **strict order**.

A strict total order completely ranks the elements of set  $S$ .

**Example:**  $<$  relation over  $\mathbb{N}_0$  gives the standard ordering  
 $0, 1, 2, 3, \dots$  of natural numbers.

German: trichotom, strenge Totalordnung

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**Example:**  $<$  relation over  $\mathbb{N}_0$  gives the standard ordering  
 $0, 1, 2, 3, \dots$  of natural numbers.

**Attention:** a non-empty strict total order is never a total order.

German: trichotom, strenge Totalordnung

# Special Elements

Special elements are defined almost as for partial orders:

Definition (Least/greatest/minimal/maximal element of a set)

Let  $\prec$  be a **strict** order over set  $S$ .

An element  $x \in S$  is the **least element** of  $S$   
if **for all**  $y \in S$  **where**  $y \neq x$  it holds that  $x \prec y$ .

It is the **greatest element** of  $S$  if **for all**  $y \in S$  **where**  $y \neq x$ ,  $y \prec x$ .

Element  $x \in S$  is a **minimal element** of  $S$   
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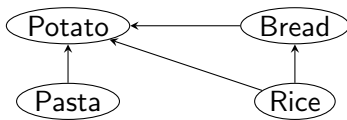
It is a **maximal element** of  $S$   
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## Special Elements – Example

Consider again the previous example:

$S = \{\text{Pasta}, \text{Potato}, \text{Bread}, \text{Rice}\}$

$\prec = \{(\text{Pasta}, \text{Potato}), (\text{Bread}, \text{Potato}),$   
 $(\text{Rice}, \text{Potato}), (\text{Rice}, \text{Bread})\}$



Is there a least and a greatest element?

Which elements are maximal or minimal?



# Questions



Questions?

# Summary

- An equivalence relation is reflexive, symmetric and transitive.

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- An equivalence relation is reflexive, symmetric and transitive.
- A partial order  $x \preceq y$  is reflexive, antisymmetric and transitive.
  - If  $x$  is the greatest element of a set  $S$ , it is greater than every element: for all  $y \in S$  it holds that  $y \preceq x$ .
  - If  $x$  is a maximal element of set  $S$  then it is not smaller than any other element  $y$ : there is no  $y \in S$  with  $x \preceq y$  and  $x \neq y$ .
  - A total order is a partial order without incomparable objects.

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  - A total order is a partial order without incomparable objects.
- A strict order is irreflexive, asymmetric and transitive.
  - Strict total orders and special elements are analogously defined as for partial orders but with a special treatment of equal elements.