

# Planning and Optimization

## B1. Transition Systems and Propositional Logic

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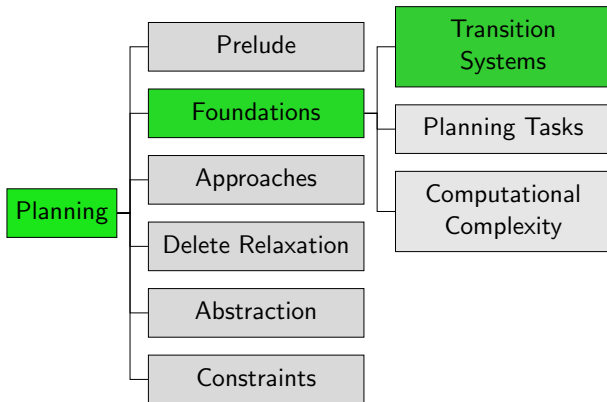
B1.1 Transition Systems

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# Content of the Course



# Next Steps

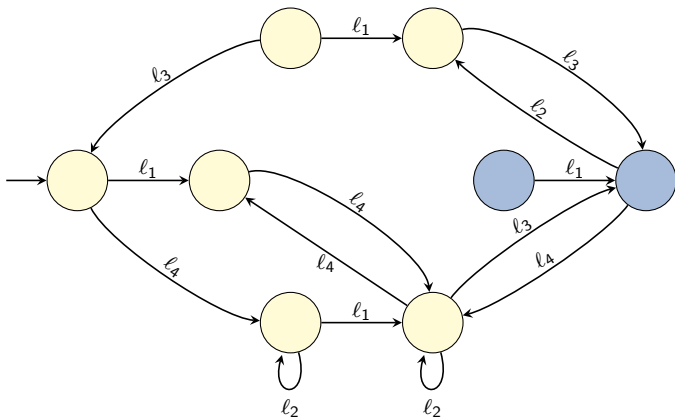
Our next steps are to formally define our problem:

- ▶ introduce a mathematical model for planning tasks:  
**transition systems**  
↪ Chapter B1
- ▶ introduce **compact representations** for planning tasks  
suitable as input for planning algorithms  
↪ Chapter B2

# B1.1 Transition Systems

# Transition System Example

Transition systems are often depicted as **directed arc-labeled graphs** with decorations to indicate the initial state and goal states.



$$c(l_1) = 1, c(l_2) = 1, c(l_3) = 5, c(l_4) = 0$$

# Transition Systems

## Definition (Transition System)

A **transition system** is a 6-tuple  $\mathcal{T} = \langle S, L, c, T, s_0, S_\star \rangle$  where

- ▶  $S$  is a finite set of **states**,
- ▶  $L$  is a finite set of (transition) **labels**,
- ▶  $c : L \rightarrow \mathbb{R}_0^+$  is a **label cost** function,
- ▶  $T \subseteq S \times L \times S$  is the **transition relation**,
- ▶  $s_0 \in S$  is the **initial state**, and
- ▶  $S_\star \subseteq S$  is the set of **goal states**.

We say that  $\mathcal{T}$  **has the transition**  $\langle s, \ell, s' \rangle$  if  $\langle s, \ell, s' \rangle \in T$ .

We also write this as  $s \xrightarrow{\ell} s'$ , or  $s \rightarrow s'$  when not interested in  $\ell$ .

**Note:** Transition systems are also called **state spaces**.

# Deterministic Transition Systems

## Definition (Deterministic Transition System)

A transition system is called **deterministic** if for all states  $s$  and all labels  $\ell$ , there is **at most one** state  $s'$  with  $s \xrightarrow{\ell} s'$ .

**Example:** previously shown transition system



# Transition System Terminology (1)

We use common terminology from graph theory:

- ▶  $s'$  **successor** of  $s$  if  $s \rightarrow s'$
- ▶  $s$  **predecessor** of  $s'$  if  $s \rightarrow s'$

## Transition System Terminology (2)

We use common terminology from graph theory:

- ▶  $s'$  **reachable** from  $s$  if there exists a sequence of transitions  $s^0 \xrightarrow{\ell_1} s^1, \dots, s^{n-1} \xrightarrow{\ell_n} s^n$  s.t.  $s^0 = s$  and  $s^n = s'$ 
  - ▶ **Note:**  $n = 0$  possible; then  $s = s'$
  - ▶  $s^0, \dots, s^n$  is called **(state) path** from  $s$  to  $s'$
  - ▶  $\ell_1, \dots, \ell_n$  is called **(label) path** from  $s$  to  $s'$
  - ▶  $s^0 \xrightarrow{\ell_1} s^1, \dots, s^{n-1} \xrightarrow{\ell_n} s^n$  is called **trace** from  $s$  to  $s'$
  - ▶ **length** of path/trace is  $n$
  - ▶ **cost** of label path/trace is  $\sum_{i=1}^n c(\ell_i)$

# Transition System Terminology (3)

We use common terminology from graph theory:

- ▶  $s'$  **reachable** (without reference state) means reachable from initial state  $s_0$
- ▶ **solution** or **goal path** from  $s$ : path from  $s$  to some  $s' \in S_*$ 
  - ▶ if  $s$  is omitted,  $s = s_0$  is implied
- ▶ transition system **solvable** if a goal path from  $s_0$  exists

## B1.2 Example: Blocks World

# Running Example: Blocks World

- ▶ Throughout the course, we occasionally use the **blocks world** domain as an example.
- ▶ In the blocks world, a number of different blocks are arranged on a table.
- ▶ Our job is to rearrange them according to a given goal.

# Blocks World Rules (1)

Location on the table does not matter.

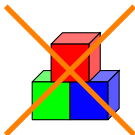


Location on a block does not matter.

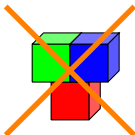


## Blocks World Rules (2)

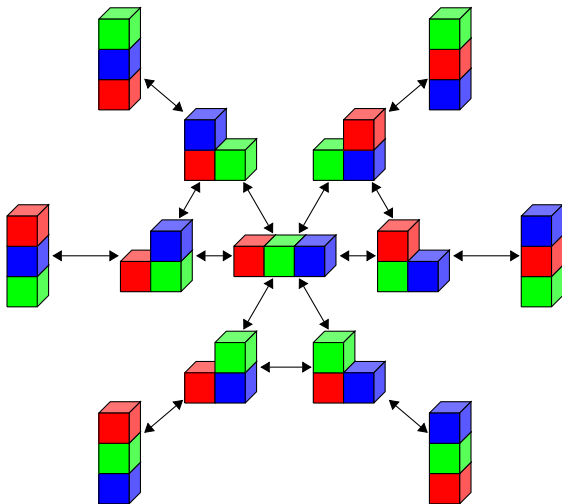
At most one block may be below a block.



At most one block may be on top of a block.



# Blocks World Transition System for Three Blocks



Labels omitted for clarity. All label costs are 1. Initial/goal states not marked.



# Blocks World Computational Properties

blocks	states	blocks	states
1	1	10	58941091
2	3	11	824073141
3	13	12	12470162233
4	73	13	202976401213
5	501	14	3535017524403
6	4051	15	65573803186921
7	37633	16	1290434218669921
8	394353	17	26846616451246353
9	4596553	18	588633468315403843

- ▶ Finding solutions is possible in linear time in the number of blocks: move everything onto the table, then construct the goal configuration.
- ▶ Finding a shortest solution is NP-complete given a compact description of the problem.

# The Need for Compact Descriptions

- ▶ We see from the blocks world example that transition systems are often **far too large** to be directly used as **inputs** to planning algorithms.
- ▶ We therefore need **compact descriptions** of transition systems.
- ▶ For this purpose, we will use **propositional logic**, which allows expressing information about  $2^n$  states as logical formulas over  $n$  **state variables**.

## B1.3 Reminder: Propositional Logic

# More on Propositional Logic

## Need to Catch Up?

- ▶ This section is a **reminder**. We assume you are already well familiar with propositional logic.
- ▶ If this is not the case, we recommend Chapters D1–D4 of the **Discrete Mathematics in Computer Science** course:  
<https://dmi.unibas.ch/en/studies/computer-science/courses-in-fall-semester-2023/lecture-discrete-mathematics-in-computer-science/>
  - ▶ Videos for these chapters are available on request.

# Syntax of Propositional Logic

## Definition (Logical Formula)

Let  $A$  be a set of **atomic propositions**.

The **logical formulas** over  $A$  are constructed by finite application of the following rules:

- ▶  $\top$  and  $\perp$  are logical formulas (**truth** and **falsity**).
- ▶ For all  $a \in A$ ,  $a$  is a logical formula (**atom**).
- ▶ If  $\varphi$  is a logical formula, then so is  $\neg\varphi$  (**negation**).
- ▶ If  $\varphi$  and  $\psi$  are logical formulas, then so are  $(\varphi \vee \psi)$  (**disjunction**) and  $(\varphi \wedge \psi)$  (**conjunction**).

# Syntactical Conventions for Propositional Logic

## Abbreviations:

- ▶  $(\varphi \rightarrow \psi)$  is short for  $(\neg\varphi \vee \psi)$  (**implication**)
- ▶  $(\varphi \leftrightarrow \psi)$  is short for  $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$  (**equijunction**)
- ▶ parentheses omitted when not necessary:
  - ▶  $(\neg)$  binds more tightly than binary connectives
  - ▶  $(\wedge)$  binds more tightly than  $(\vee)$ ,  
which binds more tightly than  $(\rightarrow)$ ,  
which binds more tightly than  $(\leftrightarrow)$

# Semantics of Propositional Logic

## Definition (Interpretation, Model)

An **interpretation** of propositions  $A$  is a function  $I : A \rightarrow \{\mathbf{T}, \mathbf{F}\}$ .

Define the notation  $I \models \varphi$  ( $I$  **satisfies**  $\varphi$ ;  $I$  is a **model** of  $\varphi$ ;  $\varphi$  is **true** under  $I$ ) for interpretations  $I$  and formulas  $\varphi$  by

- ▶  $I \models \top$
- ▶  $I \not\models \perp$
- ▶  $I \models a$             iff  $I(a) = \mathbf{T}$     (for all  $a \in A$ )
- ▶  $I \models \neg\varphi$         iff  $I \not\models \varphi$
- ▶  $I \models (\varphi \vee \psi)$     iff  $(I \models \varphi$  or  $I \models \psi)$
- ▶  $I \models (\varphi \wedge \psi)$     iff  $(I \models \varphi$  and  $I \models \psi)$

**Note:** Interpretations are also called **valuations** or **truth assignments**.

# Propositional Logic Terminology (1)

- ▶ A logical formula  $\varphi$  is **satisfiable** if there is at least one interpretation  $I$  such that  $I \models \varphi$ .
- ▶ Otherwise it is **unsatisfiable**.
- ▶ A logical formula  $\varphi$  is **valid** or a **tautology** if  $I \models \varphi$  for all interpretations  $I$ .
- ▶ A logical formula  $\psi$  is a **logical consequence** of a logical formula  $\varphi$ , written  $\varphi \models \psi$ , if  $I \models \psi$  for all interpretations  $I$  with  $I \models \varphi$ .
- ▶ Two logical formulas  $\varphi$  and  $\psi$  are **logically equivalent**, written  $\varphi \equiv \psi$ , if  $\varphi \models \psi$  and  $\psi \models \varphi$ .

**Question:** How to phrase these in terms of **models**?



## Propositional Logic Terminology (2)

- ▶ A logical formula that is a proposition  $a$  or a negated proposition  $\neg a$  for some atomic proposition  $a \in A$  is a **literal**.
- ▶ A formula that is a disjunction of literals is a **clause**. This includes **unit clauses**  $\ell$  consisting of a single literal and the **empty clause**  $\perp$  consisting of zero literals.
- ▶ A formula that is a conjunction of literals is a **monomial**. This includes **unit monomials**  $\ell$  consisting of a single literal and the **empty monomial**  $\top$  consisting of zero literals.

### Normal forms:

- ▶ negation normal form (NNF)
- ▶ conjunctive normal form (CNF)
- ▶ disjunctive normal form (DNF)

# B1.4 Summary

# Summary

- ▶ **Transition systems** are (typically huge) directed graphs that encode how the state of the world can change.
- ▶ **Propositional logic** allows us to compactly describe complex information about large sets of interpretations as **logical formulas**.