Malte Helmert, Gabriele Röger

University of Basel

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D6. Advanced Concepts in Predicate Logic and Outlook

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Free and Bound Variables

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D6.1 Free and Bound Variables

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D6.1 Free and Bound Variables

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Free and Bound Variables: Motivation

Question:

- Consider a signature with variable symbols {x₁, x₂, x₃,...} and an interpretation *I*.
- Which parts of the definition of α are relevant to decide whether $\mathcal{I}, \alpha \models (\forall x_4(\mathsf{R}(x_4, x_2) \lor (\mathsf{f}(x_3) = x_4)) \lor \exists x_3\mathsf{S}(x_3, x_2))?$
- α(x₁), α(x₅), α(x₆), α(x₇), ... are irrelevant since those variable symbols occur in no formula.
- α(x₄) also is irrelevant: the variable occurs in the formula, but all occurrences are bound by a surrounding quantifier.
- \rightarrow only assignments for free variables x_2 and x_3 relevant

German: gebundene und freie Variablen

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Free and Bound Variables

Variables of a Term

Definition (Variables of a Term)

Let t be a term. The set of variables that occur in t, written as var(t), is defined as follows:

- var(x) = {x}
 for variable symbols x
- Var(c) = ∅ for constant symbols c
- ► $var(f(t_1,...,t_k)) = var(t_1) \cup \cdots \cup var(t_k)$ for function terms

terminology: A term t with $var(t) = \emptyset$ is called ground term. German: Grundterm

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example: var(product(x, sum(k, y))) =
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Closed Formulas/Sentences

Note: Let φ be a formula and let α and β variable assignments with $\alpha(x) = \beta(x)$ for all free variables x of φ . Then $\mathcal{I}, \alpha \models \varphi$ iff $\mathcal{I}, \beta \models \varphi$.

In particular, α is completely irrelevant if $free(\varphi) = \emptyset$.

Definition (Closed Formulas/Sentences)

A formula φ without free variables (i. e., *free*(φ) = \emptyset) is called closed formula or sentence.

If φ is a sentence, then we often write $\mathcal{I} \models \varphi$ instead of $\mathcal{I}, \alpha \models \varphi$, since the definition of α does not influence whether φ is true under \mathcal{I} and α or not.

Formulas with at least one free variable are called open.

Closed formulas with no quantifiers are called ground formulas.

German: geschlossene Formel/Satz, offene Formel, Grundformel/variablenfreie Formel A. Helmert, G. Röger (University of Basel) Discrete Mathematics in Computer Science D6. Advanced Concepts in Predicate Logic and Outlook

Free and Bound Variables of a Formula

Free and Bound Variables

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Definition (Free Variables)

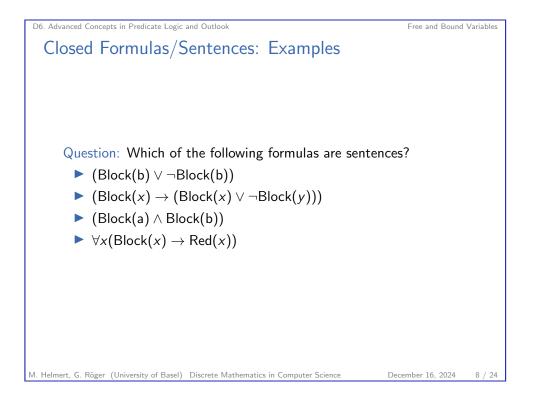
Let φ be a predicate logic formula. The set of free variables of φ , written as *free*(φ), is defined as follows:

- $free(P(t_1,\ldots,t_k)) = var(t_1) \cup \cdots \cup var(t_k)$
- $\blacktriangleright free((t_1 = t_2)) = var(t_1) \cup var(t_2)$
- free($\neg \varphi$) = free(φ)
- $\blacktriangleright \ \textit{free}((\varphi \land \psi)) = \textit{free}((\varphi \lor \psi)) = \textit{free}(\varphi) \cup \textit{free}(\psi)$
- $\models free(\forall x \varphi) = free(\exists x \varphi) = free(\varphi) \setminus \{x\}$

Example: *free*(($\forall x_4(R(x_4, x_2) \lor (f(x_3) = x_4)) \lor \exists x_3S(x_3, x_2)))$ =

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Sets of Formulas: Semantics

Definition (Satisfied/True Sets of Formulas)

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Let S be a signature, Φ a set of formulas over S, \mathcal{I} an interpretation for S and α a variable assignment for Sand the universe of \mathcal{I} .

We say that \mathcal{I} and α satisfy the formulas Φ (also: Φ is true under \mathcal{I} and α), written as: $\mathcal{I}, \alpha \models \Phi$, if $\mathcal{I}, \alpha \models \varphi$ for all $\varphi \in \Phi$.

German: \mathcal{I} und α erfüllen Φ , Φ ist wahr unter \mathcal{I} und α

We may again write $\mathcal{I} \models \Phi$ if all formulas in Φ are sentences.

Terminology for Formulas

The terminology we introduced for propositional logic equally applies to predicate logic:

- Interpretation *I* and variable assignment α form a model of the formula φ if *I*, α ⊨ φ.
- Formula φ is satisfiable if $\mathcal{I}, \alpha \models \varphi$ for at least one \mathcal{I}, α .
- ▶ Formula φ is falsifiable if $\mathcal{I}, \alpha \not\models \varphi$. for at least one \mathcal{I}, α
- Formula φ is valid if $\mathcal{I}, \alpha \models \varphi$ for all \mathcal{I}, α .
- Formula φ is unsatisfiable if $\mathcal{I}, \alpha \not\models \varphi$ for all \mathcal{I}, α .

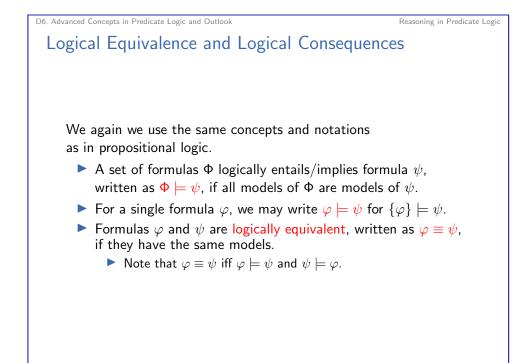
German: Modell, erfüllbar, falsifizierbar, gültig, unerfüllbar

All concepts can be used for the special case of sentences. In this case we usually omit α . Examples:

- Interpretation \mathcal{I} is a model of a sentence φ if $\mathcal{I} \models \varphi$.
- Sentence φ is unsatisfiable if $\mathcal{I} \not\models \varphi$ for all \mathcal{I} .

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Reasoning in Predicate Logic

Important Theorems about Logical Consequences

Theorem (Deduction Theorem) KB \cup { φ } $\models \psi$ *iff* KB \models ($\varphi \rightarrow \psi$)

German: Deduktionssatz

Theorem (Contraposition Theorem) KB $\cup \{\varphi\} \models \neg \psi \text{ iff } KB \cup \{\psi\} \models \neg \varphi$

German: Kontrapositionssatz

Theorem (Contradiction Theorem)

 $\mathsf{KB} \cup \{\varphi\} \text{ is unsatisfiable iff } \mathsf{KB} \models \neg \varphi$

German: Widerlegungssatz

These can be proved exactly the same way as in propositional logic.

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Normal Forms (1)

Analogously to DNF and CNF for propositional logic there are several normal forms for predicate logic, such as

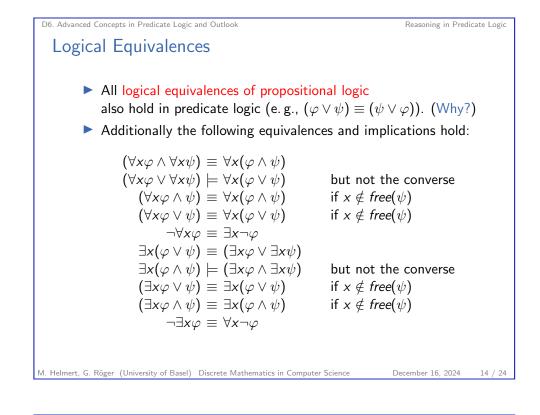
- negation normal form (NNF): negation symbols (¬) are only allowed in front of atoms
 - or identities prenex normal form:

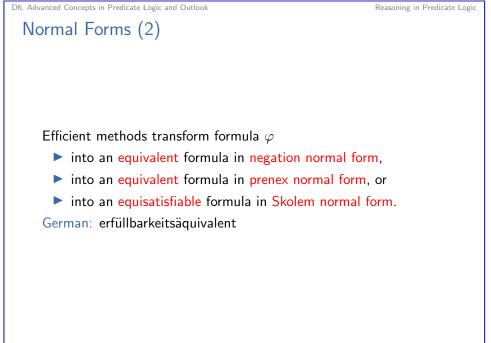
quantifiers must form the outermost part of the formula

Skolem normal form:

prenex normal form without existential quantifiers

German: Negationsnormalform, Pränexnormalform, Skolemnormalform





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Reasoning in Predicate Logic

Summary and Outlook

Inference Rules and Calculi

There exist correct and complete proof systems (calculi) for predicate logic.

- An example is the natural deduction calculus.
- ▶ This is (essentially) Gödel's Completeness Theorem (1929).
- However, one can show that correct and complete algorithms that prove that a given formula does not follow from a given set of formulas cannot exist.
- How are these statements reconcilable?

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Summary

- Predicate logic is more expressive than propositional logic and allows statements over objects and their properties.
- Objects are described by terms that are built from variable, constant and function symbols.
- Properties and relations are described by formulas that are built from predicates, quantifiers and the usual logical operators.
- Bound vs. free variables: to decide if *I*, α ⊨ φ, only free variables in α matter
- Sentences (closed formulas): formulas without free variables



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Summary

Once the basic definitions are in place, predicate logic can be developed in the same way as propositional logic:

- logical consequence
- deduction theorem etc.
- logical equivalences
- normal forms
- ▶ inference rules, proof systems, resolution

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Summary and Outlook

