Discrete Mathematics in Computer Science
D5. Syntax and Semantics of Predicate Logic

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D5.1 Syntax of Predicate Logic

D5.2 Semantics of Predicate Logic

D5.1 Syntax of Predicate Logic

Limits of Propositional Logic

Cannot be expressed well in propositional logic:

- ▶ "Everyone who does the exercises passes the exam."
- ▶ "If someone with administrator privileges presses 'delete', all data is gone."
- ▶ "Everyone has a mother."
- ▶ "If someone is the father of some person, the person is his child."

▷ need more expressive logic
  ⤳ predicate logic (a.k.a. first-order logic)

German: Prädikatenlogik (erster Stufe)

Syntax: Building Blocks

- ▶ **Signatures** define allowed symbols.
  analogy: atom set A in propositional logic
- ▶ **Terms** are associated with objects by the semantics.
  no analogy in propositional logic
- ▶ **Formulas** are associated with truth values (true or false)
  by the semantics.
  analogy: formulas in propositional logic

German: Signatur, Term, Formel

Signatures: Definition

> **Definition (Signature)**
>
> A **signature** (of predicate logic) is a 4-tuple $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$
> consisting of the following four disjoint sets:
> - ▶ a finite or countable set $\mathcal{V}$ of **variable symbols**
> - ▶ a finite or countable set $\mathcal{C}$ of **constant symbols**
> - ▶ a finite or countable set $\mathcal{F}$ of **function symbols**
> - ▶ a finite or countable set $\mathcal{P}$ of **predicate symbols**
>   (or **relation symbols**)
>
> Every function symbol $f \in \mathcal{F}$ and predicate symbol $P \in \mathcal{P}$
> has an associated **arity** $ar(f), ar(P) \in \mathbb{N}_1$ (number of arguments).

German: Variablen-, Konstanten-, Funktions-, Prädikat- und
Relationssymbole; Stelligkeit

Signatures: Terminology and Conventions

terminology:
- ▶ **k-ary** (function or predicate) symbol:
  symbol s with arity $ar(s) = k$.
- ▶ also: **unary**, **binary**, **ternary**

German: k-stellig, unär, binär, ternär

conventions (in this course):
- ▶ variable symbols written in *italics*,
  other symbols upright.
- ▶ predicate symbols begin with capital letter,
  other symbols with lower-case letters

Signatures: Examples

> **Example: Arithmetic**
> - ▶ $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
> - ▶ $\mathcal{C} = \{\text{zero}, \text{one}\}$
> - ▶ $\mathcal{F} = \{\text{sum}, \text{product}\}$
> - ▶ $\mathcal{P} = \{\text{Positive}, \text{SquareNumber}\}$
>
> $ar(\text{sum}) = ar(\text{product}) = 2$, $ar(\text{Positive}) = ar(\text{SquareNumber}) = 1$

Signatures: Examples

Example: Genealogy
- $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
- $\mathcal{C} = \{\text{roger-federer}, \text{lisa-simpson}\}$
- $\mathcal{F} = \emptyset$
- $\mathcal{P} = \{\text{Female}, \text{Male}, \text{Parent}\}$

$ar(\text{Female}) = ar(\text{Male}) = 1$, $ar(\text{Parent}) = 2$

Terms: Definition

Definition (Term)
Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.
A term (over $\mathcal{S}$) is inductively constructed
according to the following rules:
- Every variable symbol $v \in \mathcal{V}$ is a term.
- Every constant symbol $c \in \mathcal{C}$ is a term.
- If $t_1, \dots, t_k$ are terms and $f \in \mathcal{F}$ is a function symbol
  with arity k, then $f(t_1, \dots, t_k)$ is a term.

German: Term

examples:
- x_4
- lisa-simpson
- $\text{sum}(x_3, \text{product}(\text{one}, x_5))$

Formulas: Definition

Definition (Formula)
For a signature $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas
(over $\mathcal{S}$) is inductively defined as follows:

- If $t_1, \dots, t_k$ are terms (over $\mathcal{S}$) and $P \in \mathcal{P}$ is a k-ary predicate
  symbol, then the atomic formula (or the atom) $P(t_1, \dots, t_k)$
  is a formula over $\mathcal{S}$.
- If t_1 and t_2 are terms (over $\mathcal{S}$), then the identity $(t_1 = t_2)$
  is a formula over $\mathcal{S}$.
- If $x \in \mathcal{V}$ is a variable symbol and φ a formula over $\mathcal{S}$,
  then the universal quantification $\forall x\, \varphi$
  and the existential quantification $\exists x\, \varphi$ are formulas over $\mathcal{S}$.

$\dots$

German: atomare Formel, Atom, Identität,
Allquantifizierung, Existenzquantifizierung

Formulas: Definition

Definition (Formula)
For a signature $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas
(over $\mathcal{S}$) is inductively defined as follows:
$\dots$
- If φ is a formula over $\mathcal{S}$, then so is its negation $\neg\varphi$.
- If φ and ψ are formulas over $\mathcal{S}$, then so are
  the conjunction $(\varphi \wedge \psi)$ and the disjunction $(\varphi \vee \psi)$.

German: Negation, Konjunktion, Disjunktion

Formulas: Examples

> **Examples: Arithmetic and Genealogy**
> - $\text{Positive}(x_2)$
> - $\forall x\,(\neg\text{SquareNumber}(x) \vee \text{Positive}(x))$
> - $\exists x_3\,(\text{SquareNumber}(x_3) \wedge \neg\text{Positive}(x_3))$
> - $\forall x\,(x = y)$
> - $\forall x\,(\text{sum}(x, x) = \text{product}(x, \text{one}))$
> - $\forall x \exists y\,(\text{sum}(x, y) = \text{zero})$
> - $\forall x \exists y\,(\text{Parent}(y, x) \wedge \text{Female}(y))$

Terminology: The symbols $\forall$ and $\exists$ are called quantifiers.

German: Quantoren

Abbreviations and Placement of Parentheses by Convention

abbreviations:
- $(\varphi \to \psi)$ is an abbreviation for $(\neg\varphi \vee \psi)$.
- $(\varphi \leftrightarrow \psi)$ is an abbreviation for $((\varphi \to \psi) \wedge (\psi \to \varphi))$.
- Sequences of the same quantifier can be abbreviated.
  For example:
  - $\forall x \forall y \forall z\,\varphi \rightsquigarrow \forall xyz\,\varphi$
  - $\exists x \exists y \exists z\,\varphi \rightsquigarrow \exists xyz\,\varphi$
  - $\forall w \exists x \exists y \forall z\,\varphi \rightsquigarrow \forall w \exists xy \forall z\,\varphi$

placement of parentheses by convention:
- analogous to propositional logic
- quantifiers $\forall$ and $\exists$ bind more strongly than anything else.
- example: $\forall x\,P(x) \to Q(x)$ corresponds to $(\forall x\,P(x) \to Q(x))$, not $\forall x\,(P(x) \to Q(x))$.

Exercise

$\mathcal{S} = \langle \{x, y, z\}, \{c\}, \{f, g, h\}, \{Q, R, S\} \rangle$ with
$ar(f) = 3, ar(g) = ar(h) = 1, ar(Q) = 2, ar(R) = ar(S) = 1$

- $f(x, y)$
- $(g(x) = R(y))$
- $(g(x) = f(y, c, h(x)))$
- $(R(x) \wedge \forall x\,S(x))$
- $\forall c\,Q(c, x)$
- $(\forall x \exists y\,(g(x) = y) \vee (h(x) = c))$

Which expressions are syntactically correct formulas or terms for $\mathcal{S}$?
What kind of term/formula?

D5.2 Semantics of Predicate Logic

Semantics: Motivation

- interpretations in propositional logic:
  truth assignments for the propositional variables
- There are no propositional variables in predicate logic.
- instead: interpretation determines meaning
  of the constant, function and predicate symbols.
- meaning of variable symbols not determined by interpretation
  but by separate variable assignment

Interpretations and Variable Assignments

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation, Variable Assignment)

An interpretation (for $\mathcal{S}$) is a pair $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ of:

- a non-empty set U called the universe and
- a function $\cdot^{\mathcal{I}}$ that assigns a meaning to the constant,
  function, and predicate symbols:
  - $c^{\mathcal{I}} \in U$ for constant symbols $c \in \mathcal{C}$
  - $f^{\mathcal{I}} : U^k \to U$ for k-ary function symbols $f \in \mathcal{F}$
  - $P^{\mathcal{I}} \subseteq U^k$ for k-ary predicate symbols $P \in \mathcal{P}$

A variable assignment (for $\mathcal{S}$ and universe U)
is a function $\alpha : \mathcal{V} \to U$.

German: Interpretation, Universum (or Grundmenge),
          Variablenzuweisung

Interpretations and Variable Assignments: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ with $\mathcal{V} = \{x, y, z\}$,
$\mathcal{C} = \{\text{zero}, \text{one}\}$, $\mathcal{F} = \{\text{sum}, \text{product}\}$, $\mathcal{P} = \{\text{SquareNumber}\}$
$ar(\text{sum}) = ar(\text{product}) = 2$, $ar(\text{SquareNumber}) = 1$

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- $\text{zero}^{\mathcal{I}} = u_0$
- $\text{one}^{\mathcal{I}} = u_1$
- $\text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$ for all $i, j \in \{0, \ldots, 6\}$
- $\text{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$ for all $i, j \in \{0, \ldots, 6\}$
- $\text{SquareNumber}^{\mathcal{I}} = \{u_0, u_1, u_2, u_4\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

Semantics: Informally

Example: $(\forall x (\text{Block}(x) \to \text{Red}(x)) \land \text{Block}(a))$
"For all objects x: if x is a block, then x is red.
Also, the object called a is a block."

- Terms are interpreted as objects.
- Unary predicates denote properties of objects
  (to be a block, to be red, to be a square number, ...).
- General predicates denote relations between objects
  (to be someone's child, to have a common divisor, ...).
- Universally quantified formulas ("$\forall$") are true
  if they hold for every object in the universe.
- Existentially quantified formulas ("$\exists$") are true
  if they hold for at least one object in the universe.

Interpretations of Terms

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation of a Term)

Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for $\mathcal{S}$,
and let α be a variable assignment for $\mathcal{S}$ and universe U.

Let t be a term over $\mathcal{S}$.
The interpretation of t under $\mathcal{I}$ and α, written as $t^{\mathcal{I},\alpha}$,
is the element of the universe U defined as follows:

- If $t = x$ with $x \in \mathcal{V}$ (t is a variable term):
  $x^{\mathcal{I},\alpha} = \alpha(x)$
- If $t = c$ with $c \in \mathcal{C}$ (t is a constant term):
  $c^{\mathcal{I},\alpha} = c^{\mathcal{I}}$
- If $t = f(t_1, \ldots, t_k)$ (t is a function term):
  $f(t_1, \ldots, t_k)^{\mathcal{I},\alpha} = f^{\mathcal{I}}(t_1^{\mathcal{I},\alpha}, \ldots, t_k^{\mathcal{I},\alpha})$

Interpretations of Terms: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$
with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{\text{zero}, \text{one}\}$, $\mathcal{F} = \{\text{sum}, \text{product}\}$,
$ar(\text{sum}) = ar(\text{product}) = 2$

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- $\text{zero}^{\mathcal{I}} = u_0$
- $\text{one}^{\mathcal{I}} = u_1$
- $\text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$ for all $i, j \in \{0, \ldots, 6\}$
- $\text{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$ for all $i, j \in \{0, \ldots, 6\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

Interpretations of Terms: Example (ctd.)

Example (ctd.)

- $\text{zero}^{\mathcal{I},\alpha} =$

- $y^{\mathcal{I},\alpha} =$

- $\text{sum}(x, y)^{\mathcal{I},\alpha} =$

- $\text{product}(\text{one}, \text{sum}(x, \text{zero}))^{\mathcal{I},\alpha} =$

Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Formula is Satisfied or True)

Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for $\mathcal{S}$,
and let α be a variable assignment for $\mathcal{S}$ and universe U.
We say that $\mathcal{I}$ and α satisfy a predicate logic formula φ
(also: φ is true under $\mathcal{I}$ and α), written: $\mathcal{I}, \alpha \models \varphi$,
according to the following inductive rules:

$$\mathcal{I}, \alpha \models P(t_1, \ldots, t_k) \quad \text{iff} \quad \langle t_1^{\mathcal{I},\alpha}, \ldots, t_k^{\mathcal{I},\alpha} \rangle \in P^{\mathcal{I}}$$
$$\mathcal{I}, \alpha \models (t_1 = t_2) \quad \text{iff} \quad t_1^{\mathcal{I},\alpha} = t_2^{\mathcal{I},\alpha}$$
$$\mathcal{I}, \alpha \models \neg\varphi \quad \text{iff} \quad \mathcal{I}, \alpha \not\models \varphi$$
$$\mathcal{I}, \alpha \models (\varphi \wedge \psi) \quad \text{iff} \quad \mathcal{I}, \alpha \models \varphi \text{ and } \mathcal{I}, \alpha \models \psi$$
$$\mathcal{I}, \alpha \models (\varphi \vee \psi) \quad \text{iff} \quad \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi \quad \ldots$$

German: $\mathcal{I}$ und α erfüllen φ (also: φ ist wahr unter $\mathcal{I}$ und α)

Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Formula is Satisfied or True)
. . .

$$\mathcal{I}, \alpha \models \forall x \varphi \quad \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for all } u \in U$$
$$\mathcal{I}, \alpha \models \exists x \varphi \quad \text{iff } \mathcal{I}, \alpha[x := u] \models \varphi \text{ for at least one } u \in U$$

where $\alpha[x := u]$ is the same variable assignment as α,
except that it maps variable x to the value u.
Formally:
$$(\alpha[x := u])(z) = \begin{cases} u & \text{if } z = x \\ \alpha(z) & \text{if } z \neq x \end{cases}$$

Semantics: Example

Example
signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$
with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{a, b\}$, $\mathcal{F} = \emptyset$, $\mathcal{P} = \{\text{Block}, \text{Red}\}$,
$ar(\text{Block}) = ar(\text{Red}) = 1$.

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with
- $U = \{u_1, u_2, u_3, u_4, u_5\}$
- $a^{\mathcal{I}} = u_1$
- $b^{\mathcal{I}} = u_3$
- $\text{Block}^{\mathcal{I}} = \{u_1, u_2\}$
- $\text{Red}^{\mathcal{I}} = \{u_1, u_2, u_3, u_5\}$

$\alpha = \{x \mapsto u_1, y \mapsto u_2, z \mapsto u_1\}$

Semantics: Example (ctd.)

Example (ctd.)
Questions:
- $\mathcal{I}, \alpha \models (\text{Block}(b) \vee \neg\text{Block}(b))$?
- $\mathcal{I}, \alpha \models (\text{Block}(x) \rightarrow (\text{Block}(x) \vee \neg\text{Block}(y)))$?
- $\mathcal{I}, \alpha \models (\text{Block}(a) \wedge \text{Block}(b))$?
- $\mathcal{I}, \alpha \models \forall x(\text{Block}(x) \rightarrow \text{Red}(x))$?

Summary

- Predicate logic is more expressive than propositional logic and allows statements over objects and their properties.
- Objects are described by terms that are built from variable, constant and function symbols.
- Properties and relations are described by formulas that are built from predicates, quantifiers and the usual logical operators.