

Discrete Mathematics in Computer Science

D5. Syntax and Semantics of Predicate Logic

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December 9/11/16, 2024

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D5.1 Syntax of Predicate Logic

D5.2 Semantics of Predicate Logic

D5.1 Syntax of Predicate Logic

Limits of Propositional Logic

Cannot be expressed well in propositional logic:

- ▶ “Everyone who does the exercises passes the exam.”
- ▶ “If someone with administrator privileges presses ‘delete’, all data is gone.”
- ▶ “Everyone has a mother.”
- ▶ “If someone is the father of some person, the person is his child.”

▷ need more expressive logic

↪ **predicate logic** (a.k.a. first-order logic)

German: Prädikatenlogik (erster Stufe)

Syntax: Building Blocks

- ▶ **Signatures** define allowed symbols.
analogy: atom set A in propositional logic
- ▶ **Terms** are associated with objects by the semantics.
no analogy in propositional logic
- ▶ **Formulas** are associated with truth values (**true** or **false**)
by the semantics.
analogy: formulas in propositional logic

German: Signatur, Term, Formel

Signatures: Definition

Definition (Signature)

A **signature** (of predicate logic) is a 4-tuple $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ consisting of the following four disjoint sets:

- ▶ a finite or countable set \mathcal{V} of **variable symbols**
- ▶ a finite or countable set \mathcal{C} of **constant symbols**
- ▶ a finite or countable set \mathcal{F} of **function symbols**
- ▶ a finite or countable set \mathcal{P} of **predicate symbols**
(or **relation symbols**)

Every function symbol $f \in \mathcal{F}$ and predicate symbol $P \in \mathcal{P}$ has an associated **arity** $ar(f), ar(P) \in \mathbb{N}_1$ (number of arguments).

German: Variablen-, Konstanten-, Funktions-, Prädikat- und Relationssymbole; Stelligkeit

Signatures: Terminology and Conventions

terminology:

- ▶ *k*-ary (function or predicate) symbol:
symbol s with arity $ar(s) = k$.
- ▶ also: unary, binary, ternary

German: k -stellig, unär, binär, ternär

conventions (in this course):

- ▶ variable symbols written in *italics*,
other symbols upright.
- ▶ predicate symbols begin with capital letter,
other symbols with lower-case letters

Signatures: Examples

Example: Arithmetic

▶ $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$

▶ $\mathcal{C} = \{\text{zero}, \text{one}\}$

▶ $\mathcal{F} = \{\text{sum}, \text{product}\}$

▶ $\mathcal{P} = \{\text{Positive}, \text{SquareNumber}\}$

$ar(\text{sum}) = ar(\text{product}) = 2, ar(\text{Positive}) = ar(\text{SquareNumber}) = 1$

Signatures: Examples

Example: Genealogy

- ▶ $\mathcal{V} = \{x, y, z, x_1, x_2, x_3, \dots\}$
- ▶ $\mathcal{C} = \{\text{roger-federer, lisa-simpson}\}$
- ▶ $\mathcal{F} = \emptyset$
- ▶ $\mathcal{P} = \{\text{Female, Male, Parent}\}$

$ar(\text{Female}) = ar(\text{Male}) = 1, ar(\text{Parent}) = 2$

Terms: Definition

Definition (Term)

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

A **term** (over \mathcal{S}) is inductively constructed according to the following rules:

- ▶ Every variable symbol $v \in \mathcal{V}$ is a term.
- ▶ Every constant symbol $c \in \mathcal{C}$ is a term.
- ▶ If t_1, \dots, t_k are terms and $f \in \mathcal{F}$ is a function symbol with arity k , then $f(t_1, \dots, t_k)$ is a term.

German: Term

examples:

- ▶ x_4
- ▶ lisa-simpson
- ▶ $\text{sum}(x_3, \text{product}(\text{one}, x_5))$

Formulas: Definition

Definition (Formula)

For a signature $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas (over \mathcal{S}) is inductively defined as follows:

- ▶ If t_1, \dots, t_k are terms (over \mathcal{S}) and $P \in \mathcal{P}$ is a k -ary predicate symbol, then the **atomic formula** (or the **atom**) $P(t_1, \dots, t_k)$ is a formula over \mathcal{S} .
- ▶ If t_1 and t_2 are terms (over \mathcal{S}), then the **identity** $(t_1 = t_2)$ is a formula over \mathcal{S} .
- ▶ If $x \in \mathcal{V}$ is a variable symbol and φ a formula over \mathcal{S} , then the **universal quantification** $\forall x \varphi$ and the **existential quantification** $\exists x \varphi$ are formulas over \mathcal{S} .

...

German: atomare Formel, Atom, Identität,
Allquantifizierung, Existenzquantifizierung

Formulas: Definition

Definition (Formula)

For a signature $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ the set of predicate logic formulas (over \mathcal{S}) is inductively defined as follows:

...

- ▶ If φ is a formula over \mathcal{S} , then so is its **negation** $\neg\varphi$.
- ▶ If φ and ψ are formulas over \mathcal{S} , then so are the **conjunction** $(\varphi \wedge \psi)$ and the **disjunction** $(\varphi \vee \psi)$.

German: Negation, Konjunktion, Disjunktion

Formulas: Examples

Examples: Arithmetic and Genealogy

- ▶ $\text{Positive}(x_2)$
- ▶ $\forall x (\neg \text{SquareNumber}(x) \vee \text{Positive}(x))$
- ▶ $\exists x_3 (\text{SquareNumber}(x_3) \wedge \neg \text{Positive}(x_3))$
- ▶ $\forall x (x = y)$
- ▶ $\forall x (\text{sum}(x, x) = \text{product}(x, \text{one}))$
- ▶ $\forall x \exists y (\text{sum}(x, y) = \text{zero})$
- ▶ $\forall x \exists y (\text{Parent}(y, x) \wedge \text{Female}(y))$

Terminology: The symbols \forall and \exists are called **quantifiers**.

German: Quantoren

Abbreviations and Placement of Parentheses by Convention

abbreviations:

- ▶ $(\varphi \rightarrow \psi)$ is an abbreviation for $(\neg\varphi \vee \psi)$.
- ▶ $(\varphi \leftrightarrow \psi)$ is an abbreviation for $((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$.
- ▶ Sequences of the same quantifier can be abbreviated.

For example:

- ▶ $\forall x\forall y\forall z \varphi \rightsquigarrow \forall xyz \varphi$
- ▶ $\exists x\exists y\exists z \varphi \rightsquigarrow \exists xyz \varphi$
- ▶ $\forall w\exists x\exists y\forall z \varphi \rightsquigarrow \forall w\exists xy\forall z \varphi$

placement of parentheses by convention:

- ▶ analogous to propositional logic
- ▶ quantifiers \forall and \exists bind more strongly than anything else.
- ▶ **example:** $\forall x P(x) \rightarrow Q(x)$ corresponds to $(\forall x P(x) \rightarrow Q(x))$,
not $\forall x (P(x) \rightarrow Q(x))$.

Exercise

$\mathcal{S} = \langle \{x, y, z\}, \{c\}, \{f, g, h\}, \{Q, R, S\} \rangle$ with
 $ar(f) = 3, ar(g) = ar(h) = 1, ar(Q) = 2, ar(R) = ar(S) = 1$

- ▶ $f(x, y)$
- ▶ $(g(x) = R(y))$
- ▶ $(g(x) = f(y, c, h(x)))$
- ▶ $(R(x) \wedge \forall x S(x))$
- ▶ $\forall c Q(c, x)$
- ▶ $(\forall x \exists y (g(x) = y) \vee (h(x) = c))$

Which expressions are syntactically correct formulas or terms for \mathcal{S} ?
What kind of term/formula?

D5.2 Semantics of Predicate Logic

Semantics: Motivation

- ▶ interpretations in propositional logic:
truth assignments for the **propositional variables**
- ▶ There are no propositional variables in predicate logic.
- ▶ instead: interpretation determines meaning
of the **constant**, **function** and **predicate symbols**.
- ▶ meaning of **variable symbols** not determined by interpretation
but by separate **variable assignment**

Interpretations and Variable Assignments

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation, Variable Assignment)

An **interpretation** (for \mathcal{S}) is a pair $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ of:

- ▶ a non-empty set U called the **universe** and
- ▶ a function $\cdot^{\mathcal{I}}$ that assigns a meaning to the constant, function, and predicate symbols:
 - ▶ $c^{\mathcal{I}} \in U$ for constant symbols $c \in \mathcal{C}$
 - ▶ $f^{\mathcal{I}} : U^k \rightarrow U$ for k -ary function symbols $f \in \mathcal{F}$
 - ▶ $P^{\mathcal{I}} \subseteq U^k$ for k -ary predicate symbols $P \in \mathcal{P}$

A **variable assignment** (for \mathcal{S} and universe U) is a function $\alpha : \mathcal{V} \rightarrow U$.

German: Interpretation, Universum (or Grundmenge), Variablenzuweisung

Interpretations and Variable Assignments: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ with $\mathcal{V} = \{x, y, z\}$,
 $\mathcal{C} = \{\text{zero}, \text{one}\}$, $\mathcal{F} = \{\text{sum}, \text{product}\}$, $\mathcal{P} = \{\text{SquareNumber}\}$
 $ar(\text{sum}) = ar(\text{product}) = 2$, $ar(\text{SquareNumber}) = 1$

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- ▶ $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- ▶ $\text{zero}^{\mathcal{I}} = u_0$
- ▶ $\text{one}^{\mathcal{I}} = u_1$
- ▶ $\text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$
- ▶ $\text{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$
- ▶ $\text{SquareNumber}^{\mathcal{I}} = \{u_0, u_1, u_2, u_4\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

Semantics: Informally

Example: $(\forall x(\text{Block}(x) \rightarrow \text{Red}(x)) \wedge \text{Block}(a))$

“For all objects x : if x is a block, then x is red.

Also, the object called a is a block.”

- ▶ **Terms** are interpreted as **objects**.
- ▶ **Unary predicates** denote properties of objects (to be a block, to be red, to be a square number, ...).
- ▶ **General predicates** denote relations between objects (to be someone’s child, to have a common divisor, ...).
- ▶ **Universally quantified** formulas (“ \forall ”) are true if they hold for **every** object in the universe.
- ▶ **Existentially quantified** formulas (“ \exists ”) are true if they hold for **at least one** object in the universe.

Interpretations of Terms

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Interpretation of a Term)

Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for \mathcal{S} ,
and let α be a variable assignment for \mathcal{S} and universe U .

Let t be a term over \mathcal{S} .

The **interpretation of t** under \mathcal{I} and α , written as $t^{\mathcal{I}, \alpha}$,
is the element of the universe U defined as follows:

- ▶ If $t = x$ with $x \in \mathcal{V}$ (t is a **variable term**):
 $x^{\mathcal{I}, \alpha} = \alpha(x)$
- ▶ If $t = c$ with $c \in \mathcal{C}$ (t is a **constant term**):
 $c^{\mathcal{I}, \alpha} = c^{\mathcal{I}}$
- ▶ If $t = f(t_1, \dots, t_k)$ (t is a **function term**):
 $f(t_1, \dots, t_k)^{\mathcal{I}, \alpha} = f^{\mathcal{I}}(t_1^{\mathcal{I}, \alpha}, \dots, t_k^{\mathcal{I}, \alpha})$

Interpretations of Terms: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$

with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{\text{zero}, \text{one}\}$, $\mathcal{F} = \{\text{sum}, \text{product}\}$,

$ar(\text{sum}) = ar(\text{product}) = 2$

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

- ▶ $U = \{u_0, u_1, u_2, u_3, u_4, u_5, u_6\}$
- ▶ $\text{zero}^{\mathcal{I}} = u_0$
- ▶ $\text{one}^{\mathcal{I}} = u_1$
- ▶ $\text{sum}^{\mathcal{I}}(u_i, u_j) = u_{(i+j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$
- ▶ $\text{product}^{\mathcal{I}}(u_i, u_j) = u_{(i \cdot j) \bmod 7}$ for all $i, j \in \{0, \dots, 6\}$

$\alpha = \{x \mapsto u_5, y \mapsto u_5, z \mapsto u_0\}$

Interpretations of Terms: Example (ctd.)

Example (ctd.)

▶ $\text{zero}^{\mathcal{I},\alpha} =$

▶ $y^{\mathcal{I},\alpha} =$

▶ $\text{sum}(x, y)^{\mathcal{I},\alpha} =$

▶ $\text{product}(\text{one}, \text{sum}(x, \text{zero}))^{\mathcal{I},\alpha} =$

Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Formula is Satisfied or True)

Let $\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ be an interpretation for \mathcal{S} ,
and let α be a variable assignment for \mathcal{S} and universe U .
We say that \mathcal{I} and α **satisfy** a predicate logic formula φ
(also: φ is **true** under \mathcal{I} and α), written: $\mathcal{I}, \alpha \models \varphi$,
according to the following inductive rules:

$$\mathcal{I}, \alpha \models P(t_1, \dots, t_k) \quad \text{iff } \langle t_1^{\mathcal{I}, \alpha}, \dots, t_k^{\mathcal{I}, \alpha} \rangle \in P^{\mathcal{I}}$$

$$\mathcal{I}, \alpha \models (t_1 = t_2) \quad \text{iff } t_1^{\mathcal{I}, \alpha} = t_2^{\mathcal{I}, \alpha}$$

$$\mathcal{I}, \alpha \models \neg \varphi \quad \text{iff } \mathcal{I}, \alpha \not\models \varphi$$

$$\mathcal{I}, \alpha \models (\varphi \wedge \psi) \quad \text{iff } \mathcal{I}, \alpha \models \varphi \text{ and } \mathcal{I}, \alpha \models \psi$$

$$\mathcal{I}, \alpha \models (\varphi \vee \psi) \quad \text{iff } \mathcal{I}, \alpha \models \varphi \text{ or } \mathcal{I}, \alpha \models \psi \quad \dots$$

German: \mathcal{I} und α erfüllen φ (also: φ ist wahr unter \mathcal{I} und α)

Semantics of Predicate Logic Formulas

Let $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$ be a signature.

Definition (Formula is Satisfied or True)

...

$\mathcal{I}, \alpha \models \forall x \varphi$ iff $\mathcal{I}, \alpha[x := u] \models \varphi$ for all $u \in U$

$\mathcal{I}, \alpha \models \exists x \varphi$ iff $\mathcal{I}, \alpha[x := u] \models \varphi$ for at least one $u \in U$

where $\alpha[x := u]$ is the same variable assignment as α , except that it maps variable x to the value u .

Formally:

$$(\alpha[x := u])(z) = \begin{cases} u & \text{if } z = x \\ \alpha(z) & \text{if } z \neq x \end{cases}$$

Semantics: Example

Example

signature: $\mathcal{S} = \langle \mathcal{V}, \mathcal{C}, \mathcal{F}, \mathcal{P} \rangle$

with $\mathcal{V} = \{x, y, z\}$, $\mathcal{C} = \{a, b\}$, $\mathcal{F} = \emptyset$, $\mathcal{P} = \{\text{Block}, \text{Red}\}$,

$ar(\text{Block}) = ar(\text{Red}) = 1$.

$\mathcal{I} = \langle U, \cdot^{\mathcal{I}} \rangle$ with

▶ $U = \{u_1, u_2, u_3, u_4, u_5\}$

▶ $a^{\mathcal{I}} = u_1$

▶ $b^{\mathcal{I}} = u_3$

▶ $\text{Block}^{\mathcal{I}} = \{u_1, u_2\}$

▶ $\text{Red}^{\mathcal{I}} = \{u_1, u_2, u_3, u_5\}$

$\alpha = \{x \mapsto u_1, y \mapsto u_2, z \mapsto u_1\}$

Semantics: Example (ctd.)

Example (ctd.)

Questions:

- ▶ $\mathcal{I}, \alpha \models (\text{Block}(b) \vee \neg \text{Block}(b))?$
- ▶ $\mathcal{I}, \alpha \models (\text{Block}(x) \rightarrow (\text{Block}(x) \vee \neg \text{Block}(y)))?$
- ▶ $\mathcal{I}, \alpha \models (\text{Block}(a) \wedge \text{Block}(b))?$
- ▶ $\mathcal{I}, \alpha \models \forall x(\text{Block}(x) \rightarrow \text{Red}(x))?$

Summary

- ▶ **Predicate logic** is more expressive than propositional logic and allows statements over **objects** and their **properties**.
- ▶ Objects are described by **terms** that are built from variable, constant and function symbols.
- ▶ Properties and relations are described by **formulas** that are built from predicates, quantifiers and the usual logical operators.