# Discrete Mathematics in Computer Science

## D3. Normal Forms and Logical Consequence

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December 2/4, 2024

# Simplified Notation

# Parentheses

Associativity:

$$((\varphi \wedge \psi) \wedge \chi) \equiv (\varphi \wedge (\psi \wedge \chi))$$
$$((\varphi \vee \psi) \vee \chi) \equiv (\varphi \vee (\psi \vee \chi))$$

- Placement of parentheses for a conjunction of conjunctions does not influence whether an interpretation is a model.
- ditto for disjunctions of disjunctions

$\rightsquigarrow$ can omit parentheses and treat this as if parentheses placed arbitrarily

- Example: $(A_1 \wedge A_2 \wedge A_3 \wedge A_4)$ instead of $((A_1 \wedge (A_2 \wedge A_3)) \wedge A_4)$
- Example: $(\neg A \vee (B \wedge C) \vee D)$ instead of $((\neg A \vee (B \wedge C)) \vee D)$

# Parentheses

Does this mean we can always omit all parentheses and assume an arbitrary placement? $\rightsquigarrow$ No!

$$((\varphi \land \psi) \lor \chi) \not\equiv (\varphi \land (\psi \lor \chi))$$

What should $\varphi \land \psi \lor \chi$ mean?

# Placement of Parentheses by Convention

Often parentheses can be dropped in specific cases and an implicit placement is assumed:

- $\neg$ binds more strongly than $\wedge$
- $\wedge$ binds more strongly than $\vee$
- $\vee$ binds more strongly than $\rightarrow$ or $\leftrightarrow$

$\rightsquigarrow$ cf. PEMDAS/"Punkt vor Strich"

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- $\land$ binds more strongly than $\lor$
- $\lor$ binds more strongly than $\rightarrow$ or $\leftrightarrow$

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**Example**

$A \lor \neg C \land B \rightarrow A \lor \neg D$ stands for $A \lor \neg C \land B \rightarrow A \lor \neg D$

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$A \vee \neg C \wedge B \rightarrow A \vee \neg D$ stands for $((A \vee (\neg C \wedge B)) \rightarrow (A \vee \neg D))$

- often harder to read
- error-prone

$\rightsquigarrow$ not used in this course

# Short Notations for Conjunctions and Disjunctions

Short notation for addition:

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

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Analogously:

$$\bigwedge_{i=1}^{n} \varphi_i = (\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)$$
$$\bigvee_{i=1}^{n} \varphi_i = (\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n)$$

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$$\sum_{x \in \{x_1, \dots, x_n\}} x = x_1 + x_2 + \cdots + x_n$$

Analogously:

$$\bigwedge_{i=1}^{n} \varphi_i = (\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)$$

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$$\sum_{x \in \{x_1, \ldots, x_n\}} x = x_1 + x_2 + \cdots + x_n$$

Analogously (possible because of commutativity of $\wedge$ and $\vee$):

$$\bigwedge_{i=1}^{n} \varphi_i = (\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)$$
$$\bigvee_{i=1}^{n} \varphi_i = (\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n)$$
$$\bigwedge_{\varphi \in X} \varphi = (\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)$$
$$\bigvee_{\varphi \in X} \varphi = (\varphi_1 \vee \varphi_2 \vee \cdots \vee \varphi_n)$$
$$\text{for } X = \{\varphi_1, \ldots, \varphi_n\}$$

# Short Notation: Corner Cases

Is $\mathcal{I} \models \psi$ true for

$$\psi = \bigwedge_{\varphi \in X} \varphi \text{ and } \psi = \bigvee_{\varphi \in X} \varphi$$

if $X = \emptyset$ or $X = \{\chi\}$?

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if $X = \emptyset$ or $X = \{\chi\}$?

convention:

- $\bigwedge_{\varphi \in \emptyset} \varphi$ is a tautology.
- $\bigvee_{\varphi \in \emptyset} \varphi$ is unsatisfiable.
- $\bigwedge_{\varphi \in \{\chi\}} \varphi = \bigvee_{\varphi \in \{\chi\}} \varphi = \chi$

$\rightsquigarrow$ Why?

# Exercise

Express $\bigwedge_{i=1}^{2} \bigvee_{j=1}^{3} \varphi_{ij}$ without $\bigwedge$ and $\bigvee$.

# Normal Forms

# Why Normal Forms?

- A normal form is a representation with certain syntactic restrictions.
- condition for reasonable normal form: every formula must have a logically equivalent formula in normal form
- advantages:
  - can restrict proofs to formulas in normal form
  - can define algorithms to work only for formulas in normal form

German: Normalform

# Negation Normal Form

### Definition (Negation Normal Form)

A formula is in negation normal form (NNF)
if it does not contain the abbreviations $\rightarrow$ and $\leftrightarrow$
and if it contains no negation symbols except possibly
directly in front of atomic propositions.

German: Negationsnormalform

### Example

$((\neg P \lor (R \land Q)) \land (P \lor \neg S))$ is in NNF.
$(P \land \neg(Q \lor R))$ is not in NNF.

# Construction of NNF

## Algorithm to Construct NNF

1. Replace abbreviation $\leftrightarrow$ by its definition (($\leftrightarrow$)-elimination).
   $\rightsquigarrow$ formula structure: only $\neg$, $\vee$, $\wedge$, $\rightarrow$
2. Replace abbreviation $\rightarrow$ by its definition (($\rightarrow$)-elimination).
   $\rightsquigarrow$ formula structure: only $\neg$, $\vee$, $\wedge$
3. Repeatedly apply double negation and De Morgan rules until no rules match any more ("move negations inside"):
   - Replace $\neg\neg\varphi$ by $\varphi$.
   - Replace $\neg(\varphi \wedge \psi)$ by $(\neg\varphi \vee \neg\psi)$.
   - Replace $\neg(\varphi \vee \psi)$ by $(\neg\varphi \wedge \neg\psi)$.

   $\rightsquigarrow$ formula structure: only atoms, negated atoms, $\vee$, $\wedge$

# Constructing NNF: Example

## Construction of Negation Normal Form

Given: $\varphi = (((\mathsf{P} \wedge \neg \mathsf{Q}) \vee \mathsf{R}) \rightarrow (\mathsf{P} \vee \neg(\mathsf{S} \vee \mathsf{T})))$

# Constructing NNF: Example

## Construction of Negation Normal Form

Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

$$\varphi \equiv (\neg((P \wedge \neg Q) \vee R) \vee P \vee \neg(S \vee T)) \qquad \text{[Step 2]}$$

# Constructing NNF: Example

## Construction of Negation Normal Form

Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

$$\begin{aligned} \varphi &\equiv (\neg((P \wedge \neg Q) \vee R) \vee P \vee \neg(S \vee T)) && \text{[Step 2]} \\ &\equiv ((\neg(P \wedge \neg Q) \wedge \neg R) \vee P \vee \neg(S \vee T)) && \text{[Step 3]} \end{aligned}$$

# Constructing NNF: Example

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Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

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\begin{aligned}
\varphi &\equiv (\neg((P \wedge \neg Q) \vee R) \vee P \vee \neg(S \vee T)) && \text{[Step 2]} \\
&\equiv ((\neg(P \wedge \neg Q) \wedge \neg R) \vee P \vee \neg(S \vee T)) && \text{[Step 3]} \\
&\equiv (((\neg P \vee \neg\neg Q) \wedge \neg R) \vee P \vee \neg(S \vee T)) && \text{[Step 3]}
\end{aligned}
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&\equiv (((\neg P \vee Q) \wedge \neg R) \vee P \vee \neg(S \vee T)) && \text{[Step 3]}
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&\equiv (((\neg P \vee Q) \wedge \neg R) \vee P \vee \neg(S \vee T)) && \text{[Step 3]} \\
&\equiv (((\neg P \vee Q) \wedge \neg R) \vee P \vee (\neg S \wedge \neg T)) && \text{[Step 3]}
\end{aligned}
$$

# Literals, Clauses and Monomials

- A literal is an atomic proposition
  or the negation of an atomic proposition (e. g., $A$ and $\neg A$).
- A clause is a disjunction of literals
  (e. g., $(Q \lor \neg P \lor \neg S \lor R)$).
- A monomial is a conjunction of literals
  (e. g., $(Q \land \neg P \land \neg S \land R)$).

The terms clause and monomial are also used for the corner case with only one literal.

German: Literal, Klausel, Monom

# Terminology: Examples

## Examples

- $(\neg Q \wedge R)$
- $(P \vee \neg Q)$
- $((P \vee \neg Q) \wedge P)$
- $\neg P$
- $(P \rightarrow Q)$

- $(P \vee P)$
- $\neg\neg P$

# Terminology: Examples

## Examples

- $(\neg Q \wedge R)$ is a monomial
- $(P \vee \neg Q)$
- $((P \vee \neg Q) \wedge P)$
- $\neg P$
- $(P \rightarrow Q)$

- $(P \vee P)$
- $\neg\neg P$

# Terminology: Examples

## Examples

- $(\neg Q \wedge R)$ is a monomial
- $(P \vee \neg Q)$ is a clause
- $((P \vee \neg Q) \wedge P)$
- $\neg P$
- $(P \rightarrow Q)$
- $(P \vee P)$
- $\neg\neg P$

# Terminology: Examples

### Examples

- $(\neg Q \wedge R)$ is a monomial
- $(P \vee \neg Q)$ is a clause
- $((P \vee \neg Q) \wedge P)$ is neither literal nor clause nor monomial
- $\neg P$
- $(P \rightarrow Q)$

- $(P \vee P)$
- $\neg\neg P$

# Terminology: Examples

## Examples

- $(\neg Q \land R)$ is a monomial
- $(P \lor \neg Q)$ is a clause
- $((P \lor \neg Q) \land P)$ is neither literal nor clause nor monomial
- $\neg P$ is a literal, a clause and a monomial
- $(P \rightarrow Q)$
- $(P \lor P)$
- $\neg\neg P$

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- $\neg P$ is a literal, a clause and a monomial
- $(P \rightarrow Q)$ is neither literal nor clause nor monomial
  (but $(\neg P \vee Q)$ is a clause!)
- $(P \vee P)$
- $\neg\neg P$

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- $(P \rightarrow Q)$ is neither literal nor clause nor monomial
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- $(P \vee P)$ is a clause, but not a literal or monomial
- $\neg\neg P$

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  (but $(\neg P \vee Q)$ is a clause!)
- $(P \vee P)$ is a clause, but not a literal or monomial
- $\neg\neg P$ is neither literal nor clause nor monomial

# Conjunctive Normal Form

## Definition (Conjunctive Normal Form)

A formula is in conjunctive normal form (CNF)
if it is a conjunction of clauses, i. e., if it has the form

$$\bigwedge_{i=1}^{n} \bigvee_{j=1}^{m_i} L_{ij}$$

with $n, m_i > 0$ (for $1 \leq i \leq n$), where the $L_{ij}$ are literals.

German: konjunktive Normalform (KNF)

## Example

$((\neg P \vee Q) \wedge R \wedge (P \vee \neg S))$ is in CNF.

# Disjunctive Normal Form

**Definition (Disjunctive Normal Form)**

A formula is in disjunctive normal form (DNF)
if it is a disjunction of monomials, i. e., if it has the form

$$\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m_i} L_{ij}$$

with $n, m_i > 0$ (for $1 \leq i \leq n$), where the $L_{ij}$ are literals.

German: disjunktive Normalform (DNF)

**Example**

$((\neg P \land Q) \lor R \lor (P \land \neg S))$ is in DNF.

# NNF, CNF and DNF: Examples

Which of the following formulas are in NNF?
Which are in CNF? Which are in DNF?

- $((P \vee \neg Q) \wedge P)$
- $((R \vee Q) \wedge P \wedge (R \vee S))$
- $(P \vee (\neg Q \wedge R))$
- $(P \vee \neg\neg Q)$
- $(P \rightarrow \neg Q)$

- $((P \vee \neg Q) \rightarrow P)$
- $P$

# NNF, CNF and DNF: Examples

Which of the following formulas are in NNF?
Which are in CNF? Which are in DNF?

- $((P \vee \neg Q) \wedge P)$ is in NNF and CNF
- $((R \vee Q) \wedge P \wedge (R \vee S))$
- $(P \vee (\neg Q \wedge R))$
- $(P \vee \neg\neg Q)$
- $(P \rightarrow \neg Q)$

- $((P \vee \neg Q) \rightarrow P)$
- $P$

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Which of the following formulas are in NNF?
Which are in CNF? Which are in DNF?

- $((P \vee \neg Q) \wedge P)$ is in NNF and CNF
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- $(P \vee (\neg Q \wedge R))$
- $(P \vee \neg\neg Q)$
- $(P \rightarrow \neg Q)$

- $((P \vee \neg Q) \rightarrow P)$
- $P$

# NNF, CNF and DNF: Examples

Which of the following formulas are in NNF?
Which are in CNF? Which are in DNF?

- $((P \lor \neg Q) \land P)$ is in NNF and CNF
- $((R \lor Q) \land P \land (R \lor S))$ is in NNF and CNF
- $(P \lor (\neg Q \land R))$ is in NNF and DNF
- $(P \lor \neg\neg Q)$
- $(P \to \neg Q)$

- $((P \lor \neg Q) \to P)$
- $P$

# NNF, CNF and DNF: Examples

Which of the following formulas are in NNF?
Which are in CNF? Which are in DNF?

- $((P \vee \neg Q) \wedge P)$ is in NNF and CNF
- $((R \vee Q) \wedge P \wedge (R \vee S))$ is in NNF and CNF
- $(P \vee (\neg Q \wedge R))$ is in NNF and DNF
- $(P \vee \neg\neg Q)$ is in none of the normal forms
- $(P \rightarrow \neg Q)$

- $((P \vee \neg Q) \rightarrow P)$
- $P$

# NNF, CNF and DNF: Examples

Which of the following formulas are in NNF?
Which are in CNF? Which are in DNF?

- $((P \lor \neg Q) \land P)$ is in NNF and CNF
- $((R \lor Q) \land P \land (R \lor S))$ is in NNF and CNF
- $(P \lor (\neg Q \land R))$ is in NNF and DNF
- $(P \lor \neg\neg Q)$ is in none of the normal forms
- $(P \to \neg Q)$ is in none of the normal forms, but is in all three after expanding $\to$
- $((P \lor \neg Q) \to P)$
- $P$

# NNF, CNF and DNF: Examples

Which of the following formulas are in NNF?
Which are in CNF? Which are in DNF?

- $((P \vee \neg Q) \wedge P)$ is in NNF and CNF
- $((R \vee Q) \wedge P \wedge (R \vee S))$ is in NNF and CNF
- $(P \vee (\neg Q \wedge R))$ is in NNF and DNF
- $(P \vee \neg\neg Q)$ is in none of the normal forms
- $(P \rightarrow \neg Q)$ is in none of the normal forms,
  but is in all three after expanding $\rightarrow$
- $((P \vee \neg Q) \rightarrow P)$ is in none of the normal forms
- $P$

# NNF, CNF and DNF: Examples

Which of the following formulas are in NNF?
Which are in CNF? Which are in DNF?

- $((P \lor \neg Q) \land P)$ is in NNF and CNF
- $((R \lor Q) \land P \land (R \lor S))$ is in NNF and CNF
- $(P \lor (\neg Q \land R))$ is in NNF and DNF
- $(P \lor \neg\neg Q)$ is in none of the normal forms
- $(P \to \neg Q)$ is in none of the normal forms,
  but is in all three after expanding $\to$
- $((P \lor \neg Q) \to P)$ is in none of the normal forms
- $P$ is in NNF, CNF and DNF

# Construction of CNF (and DNF)

## Algorithm to Construct CNF

First, convert to NNF (steps 1–3).

$\rightsquigarrow$ formula structure: only literals, $\vee$, $\wedge$

4. Repeatedly apply distributivity or commutativity + distributivity to distribute $\vee$ over $\wedge$:
   - Replace $(\varphi \vee (\psi \wedge \chi))$ by $((\varphi \vee \psi) \wedge (\varphi \vee \chi))$.
   - Replace $((\psi \wedge \chi) \vee \varphi)$ by $((\psi \vee \varphi) \wedge (\chi \vee \varphi))$.

   $\rightsquigarrow$ formula structure: CNF
5. optionally: Simplify the formula at the end or at intermediate steps (e. g., with idempotence).

Note: For DNF, swap the roles of $\wedge$ and $\vee$ in Step 4.

# Constructing CNF: Example

### Construction of Conjunctive Normal Form

Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

# Constructing CNF: Example

### Construction of Conjunctive Normal Form

Given: $\varphi = (((P \land \neg Q) \lor R) \to (P \lor \neg(S \lor T)))$

$$\varphi \equiv (((\neg P \lor Q) \land \neg R) \lor P \lor (\neg S \land \neg T)) \text{ [to NNF]}$$

# Constructing CNF: Example

**Construction of Conjunctive Normal Form**

Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

$$\begin{aligned} \varphi &\equiv (((\neg P \vee Q) \wedge \neg R) \vee P \vee (\neg S \wedge \neg T)) && \text{[to NNF]} \\ &\equiv ((\neg P \vee Q \vee P \vee (\neg S \wedge \neg T)) \wedge \\ &\quad (\neg R \vee P \vee (\neg S \wedge \neg T))) && \text{[Step 4]} \end{aligned}$$

# Constructing CNF: Example

## Construction of Conjunctive Normal Form

Given: $\varphi = (((\mathsf{P} \wedge \neg \mathsf{Q}) \vee \mathsf{R}) \rightarrow (\mathsf{P} \vee \neg(\mathsf{S} \vee \mathsf{T})))$

$$
\begin{aligned}
\varphi &\equiv (((\neg \mathsf{P} \vee \mathsf{Q}) \wedge \neg \mathsf{R}) \vee \mathsf{P} \vee (\neg \mathsf{S} \wedge \neg \mathsf{T})) && \text{[to NNF]} \\
&\equiv ((\neg \mathsf{P} \vee \mathsf{Q} \vee \mathsf{P} \vee (\neg \mathsf{S} \wedge \neg \mathsf{T})) \wedge \\
&\quad (\neg \mathsf{R} \vee \mathsf{P} \vee (\neg \mathsf{S} \wedge \neg \mathsf{T}))) && \text{[Step 4]} \\
&\equiv (\neg \mathsf{R} \vee \mathsf{P} \vee (\neg \mathsf{S} \wedge \neg \mathsf{T})) && \text{[Step 5]}
\end{aligned}
$$

# Constructing CNF: Example

## Construction of Conjunctive Normal Form

Given: $\varphi = (((P \wedge \neg Q) \vee R) \rightarrow (P \vee \neg(S \vee T)))$

$$
\begin{aligned}
\varphi &\equiv (((\neg P \vee Q) \wedge \neg R) \vee P \vee (\neg S \wedge \neg T)) && \text{[to NNF]} \\
&\equiv ((\neg P \vee Q \vee P \vee (\neg S \wedge \neg T)) \wedge \\
&\quad\; (\neg R \vee P \vee (\neg S \wedge \neg T))) && \text{[Step 4]} \\
&\equiv (\neg R \vee P \vee (\neg S \wedge \neg T)) && \text{[Step 5]} \\
&\equiv ((\neg R \vee P \vee \neg S) \wedge (\neg R \vee P \vee \neg T)) && \text{[Step 4]}
\end{aligned}
$$

# Construct DNF: Example

**Construction of Disjunctive Normal Form**

Given: $\varphi = (((P \land \neg Q) \lor R) \rightarrow (P \lor \neg(S \lor T)))$

$$\begin{aligned} \varphi &\equiv (((\neg P \lor Q) \land \neg R) \lor P \lor (\neg S \land \neg T)) && \text{[to NNF]} \\ &\equiv ((\neg P \land \neg R) \lor (Q \land \neg R) \lor P \lor (\neg S \land \neg T)) && \text{[Step 4]} \end{aligned}$$

# Existence of an Equivalent Formula in Normal Form

**Theorem**

*For every formula $\varphi$ there is a logically equivalent formula in NNF, a logically equivalent formula in CNF and a logically equivalent formula in DNF.*

- "There is a" always means "there is at least one". Otherwise we would write "there is exactly one".
- Intuition: algorithms to construct normal forms work with any given formula and only use equivalence rewriting.
- actual proof would use induction over structure of formula

# Size of Normal Forms

- In the worst case, a logically equivalent formula in CNF or DNF can be exponentially larger than the original formula.
- Example: for $(x_1 \vee y_1) \wedge \cdots \wedge (x_n \vee y_n)$ there is no smaller logically equivalent formula in DNF than:

$$\bigvee_{S \in \mathcal{P}(\{1,\ldots,n\})} \left( \bigwedge_{i \in S} x_i \wedge \bigwedge_{i \in \{1,\ldots,n\} \setminus S} y_i \right)$$

- As a consequence, the construction of the CNF/DNF formula can take exponential time.
- For NNF, we can generate an equivalent formula in linear time if the original formula does not use $\leftrightarrow$.

# More Theorems

**Theorem**

*A formula in CNF is a tautology iff every clause is a tautology.*

**Theorem**

*A formula in DNF is satisfiable iff at least one of its monomials is satisfiable.*

$\rightsquigarrow$ both proved easily with semantics of propositional logic

# Knowledge Bases

# Knowledge Bases: Example

If not DrinkBeer, then EatFish.
If EatFish and DrinkBeer,
then not EatIceCream.
If EatIceCream or not DrinkBeer,
then not EatFish.

$$\begin{aligned}\mathsf{KB} = \{&(\neg\mathsf{DrinkBeer} \rightarrow \mathsf{EatFish}),\\ &((\mathsf{EatFish} \wedge \mathsf{DrinkBeer}) \rightarrow \neg\mathsf{EatIceCream}),\\ &((\mathsf{EatIceCream} \vee \neg\mathsf{DrinkBeer}) \rightarrow \neg\mathsf{EatFish})\}\end{aligned}$$

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

# Models for Sets of Formulas

**Definition (Model for Knowledge Base)**

Let KB be a knowledge base over $A$,
i. e., a set of propositional formulas over $A$.

A truth assignment $\mathcal{I}$ for $A$ is a model for KB (written: $\mathcal{I} \models \text{KB}$)
if $\mathcal{I}$ is a model for every formula $\varphi \in \text{KB}$.

German: Wissensbasis, Modell

# Properties of Sets of Formulas

A knowledge base KB is

- satisfiable if KB has at least one model
- unsatisfiable if KB is not satisfiable
- valid (or a tautology) if every interpretation is a model for KB
- falsifiable if KB is no tautology

German: erfüllbar, unerfüllbar, gültig, gültig/eine Tautologie, falsifizierbar

# Example I

Which of the properties does $\mathsf{KB} = \{(\mathsf{A} \land \neg \mathsf{B}), \neg(\mathsf{B} \lor \mathsf{A})\}$ have?

# Example I

Which of the properties does $\mathsf{KB} = \{(\mathsf{A} \wedge \neg \mathsf{B}), \neg(\mathsf{B} \vee \mathsf{A})\}$ have?

KB is unsatisfiable:
For every model $\mathcal{I}$ with $\mathcal{I} \models (\mathsf{A} \wedge \neg \mathsf{B})$ we have $\mathcal{I}(\mathsf{A}) = 1$.
This means $\mathcal{I} \models (\mathsf{B} \vee \mathsf{A})$ and thus $\mathcal{I} \not\models \neg(\mathsf{B} \vee \mathsf{A})$.

This directly implies that KB is falsifiable, not satisfiable and no tautology.

# Example II

Which of the properties does

$$\begin{aligned}\mathsf{KB} = \{&(\neg \mathsf{DrinkBeer} \rightarrow \mathsf{EatFish}),\\ &((\mathsf{EatFish} \wedge \mathsf{DrinkBeer}) \rightarrow \neg \mathsf{EatIceCream}),\\ &((\mathsf{EatIceCream} \vee \neg \mathsf{DrinkBeer}) \rightarrow \neg \mathsf{EatFish})\}\text{ have?}\end{aligned}$$

# Example II

Which of the properties does

$$\begin{aligned}\mathsf{KB} = \{&(\neg\mathsf{DrinkBeer} \rightarrow \mathsf{EatFish}),\\ &((\mathsf{EatFish} \wedge \mathsf{DrinkBeer}) \rightarrow \neg\mathsf{EatIceCream}),\\ &((\mathsf{EatIceCream} \vee \neg\mathsf{DrinkBeer}) \rightarrow \neg\mathsf{EatFish})\} \text{ have?}\end{aligned}$$

- satisfiable, e. g. with
  $\mathcal{I} = \{\mathsf{EatFish} \mapsto 1, \mathsf{DrinkBeer} \mapsto 1, \mathsf{EatIceCream} \mapsto 0\}$
- thus not unsatisfiable
- falsifiable, e. g. with
  $\mathcal{I} = \{\mathsf{EatFish} \mapsto 0, \mathsf{DrinkBeer} \mapsto 0, \mathsf{EatIceCream} \mapsto 1\}$
- thus not valid

# Logical Consequences

# Logical Consequences: Motivation

What's the secret of your long life?

I am on a strict diet: If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

Claim: the woman drinks beer to every meal.

How can we prove this?

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut/FreeDigitalPhotos.net

# Logical Consequences

**Definition (Logical Consequence)**

Let KB be a set of formulas and $\varphi$ a formula.

We say that KB logically implies $\varphi$ (written as KB $\models \varphi$) if all models of KB are also models of $\varphi$.

also: KB logically entails $\varphi$, $\varphi$ logically follows from KB, $\varphi$ is a logical consequence of KB

German: KB impliziert $\varphi$ logisch, $\varphi$ folgt logisch aus KB, $\varphi$ ist logische Konsequenz von KB

Attention: the symbol $\models$ is "overloaded": KB $\models \varphi$ vs. $\mathcal{I} \models \varphi$.

What if KB is unsatisfiable or the empty set?

# Logical Consequences: Example

Let $\varphi = \mathsf{DrinkBeer}$ and

$$
\begin{aligned}
\mathsf{KB} = \{ & (\neg \mathsf{DrinkBeer} \rightarrow \mathsf{EatFish}), \\
& ((\mathsf{EatFish} \wedge \mathsf{DrinkBeer}) \rightarrow \neg \mathsf{EatIceCream}), \\
& ((\mathsf{EatIceCream} \vee \neg \mathsf{DrinkBeer}) \rightarrow \neg \mathsf{EatFish})\}.
\end{aligned}
$$

Show: $\mathsf{KB} \models \varphi$

# Logical Consequences: Example

Let $\varphi = \mathsf{DrinkBeer}$ and

$$\begin{aligned} \mathsf{KB} = \{ & (\neg \mathsf{DrinkBeer} \rightarrow \mathsf{EatFish}), \\ & ((\mathsf{EatFish} \wedge \mathsf{DrinkBeer}) \rightarrow \neg \mathsf{EatIceCream}), \\ & ((\mathsf{EatIceCream} \vee \neg \mathsf{DrinkBeer}) \rightarrow \neg \mathsf{EatFish}) \}. \end{aligned}$$

Show: $\mathsf{KB} \models \varphi$

**Proof sketch.**

Proof by contradiction: assume $\mathcal{I} \models \mathsf{KB}$, but $\mathcal{I} \not\models \mathsf{DrinkBeer}$.
Then it follows that $\mathcal{I} \models \neg \mathsf{DrinkBeer}$.
Because $\mathcal{I}$ is a model of KB, we also have
$\mathcal{I} \models (\neg \mathsf{DrinkBeer} \rightarrow \mathsf{EatFish})$ and thus $\mathcal{I} \models \mathsf{EatFish}$. (Why?)
With an analogous argumentation starting from
$\mathcal{I} \models ((\mathsf{EatIceCream} \vee \neg \mathsf{DrinkBeer}) \rightarrow \neg \mathsf{EatFish})$
we get $\mathcal{I} \models \neg \mathsf{EatFish}$ and thus $\mathcal{I} \not\models \mathsf{EatFish}$. $\leadsto$ Contradiction!

# Important Theorems about Logical Consequences

**Theorem (Deduction Theorem)**

$\mathsf{KB} \cup \{\varphi\} \models \psi$ *iff* $\mathsf{KB} \models (\varphi \rightarrow \psi)$

German: Deduktionssatz

**Theorem (Contraposition Theorem)**

$\mathsf{KB} \cup \{\varphi\} \models \neg\psi$ *iff* $\mathsf{KB} \cup \{\psi\} \models \neg\varphi$

German: Kontrapositionssatz

**Theorem (Contradiction Theorem)**

$\mathsf{KB} \cup \{\varphi\}$ *is unsatisfiable iff* $\mathsf{KB} \models \neg\varphi$

German: Widerlegungssatz

(without proof)