# Discrete Mathematics in Computer Science

## D2. Properties of Formulas and Equivalences

Malte Helmert, Gabriele Röger

University of Basel

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D2.1 Properties of Propositional Formulas

D2.2 Equivalences





# D2.1 Properties of Propositional Formulas





## The Story So Far

- propositional logic based on atomic propositions
- syntax: which formulas are well-formed?
- semantics: when is a formula true?
- interpretations: important basis of semantics

## Reminder: Syntax of Propositional Logic

**Definition (Syntax of Propositional Logic)**

Let $A$ be a set of atomic propositions. The set of propositional formulas (over $A$) is inductively defined as follows:

- Every atom $a \in A$ is a propositional formula over $A$.
- If $\varphi$ is a propositional formula over $A$, then so is its negation $\neg\varphi$.
- If $\varphi$ and $\psi$ are propositional formulas over $A$, then so is the conjunction $(\varphi \land \psi)$.
- If $\varphi$ and $\psi$ are propositional formulas over $A$, then so is the disjunction $(\varphi \lor \psi)$.

The implication $(\varphi \to \psi)$ is an abbreviation for $(\neg\varphi \lor \psi)$.
The biconditional $(\varphi \leftrightarrow \psi)$ is an abbrev. for $((\varphi \to \psi) \land (\psi \to \varphi))$.





## Reminder: Semantics of Propositional Logic

**Definition (Semantics of Propositional Logic)**

A truth assignment (or interpretation) for a set of atomic propositions $A$ is a function $\mathcal{I} : A \to \{0, 1\}$.

A propositional formula $\varphi$ (over $A$) holds under $\mathcal{I}$ (written as $\mathcal{I} \models \varphi$) according to the following definition:

$$\begin{array}{lll l}
\mathcal{I} \models a & \text{iff} & \mathcal{I}(a) = 1 & (\text{for } a \in A)\\
\mathcal{I} \models \neg\varphi & \text{iff} & \text{not } \mathcal{I} \models \varphi & \\
\mathcal{I} \models (\varphi \land \psi) & \text{iff} & \mathcal{I} \models \varphi \text{ and } \mathcal{I} \models \psi & \\
\mathcal{I} \models (\varphi \lor \psi) & \text{iff} & \mathcal{I} \models \varphi \text{ or } \mathcal{I} \models \psi &
\end{array}$$





## Properties of Propositional Formulas

A propositional formula $\varphi$ is

- satisfiable if $\varphi$ has at least one model
- unsatisfiable if $\varphi$ is not satisfiable
- valid (or a tautology) if $\varphi$ is true under every interpretation
- falsifiable if $\varphi$ is no tautology

German: erfüllbar, unerfüllbar, allgemeingültig/eine Tautologie, falsifizierbar





## Examples

How can we show that a formula has one of these properties?

- Show that $(\mathrm{A} \land \mathrm{B})$ is satisfiable.
  $\mathcal{I} = \{\mathrm{A} \mapsto 1, \mathrm{B} \mapsto 1\}$ (+ simple proof that $\mathcal{I} \models (\mathrm{A} \land \mathrm{B})$)
- Show that $(\mathrm{A} \land \mathrm{B})$ is falsifiable.
  $\mathcal{I} = \{\mathrm{A} \mapsto 0, \mathrm{B} \mapsto 1\}$ (+ simple proof that $\mathcal{I} \not\models (\mathrm{A} \land \mathrm{B})$)
- Show that $(\mathrm{A} \land \mathrm{B})$ is not valid.
  Follows directly from falsifiability.
- Show that $(\mathrm{A} \land \mathrm{B})$ is not unsatisfiable.
  Follows directly from satisfiability.

So far all proofs by specifying one interpretation.

How to prove that a given formula is valid/unsatisfiable/not satisfiable/not falsifiable?

$\rightsquigarrow$ must consider all possible interpretations

# Truth Tables

Evaluate for all possible interpretations
if they are models of the considered formula.

| $\mathcal{I}(A)$ | $\mathcal{I} \models \neg A$ |
|---|---|
| 0 | Yes |
| 1 | No |

| $\mathcal{I}(A)$ | $\mathcal{I}(B)$ | $\mathcal{I} \models (A \land B)$ |
|---|---|---|
| 0 | 0 | No |
| 0 | 1 | No |
| 1 | 0 | No |
| 1 | 1 | Yes |

| $\mathcal{I}(A)$ | $\mathcal{I}(B)$ | $\mathcal{I} \models (A \lor B)$ |
|---|---|---|
| 0 | 0 | No |
| 0 | 1 | Yes |
| 1 | 0 | Yes |
| 1 | 1 | Yes |





# Truth Tables in General

Similarly in the case where we consider a formula whose building blocks are themselves arbitrary unspecified formulas:

| $\mathcal{I} \models \varphi$ | $\mathcal{I} \models \psi$ | $\mathcal{I} \models (\varphi \land \psi)$ |
|---|---|---|
| No | No | No |
| No | Yes | No |
| Yes | No | No |
| Yes | Yes | Yes |





# Truth Tables for Properties of Formulas

Is $\varphi = ((A \rightarrow B) \lor (\neg B \rightarrow A))$ valid, unsatisfiable, ...?

| $\mathcal{I}(A)$ | $\mathcal{I}(B)$ | $\mathcal{I} \models \neg B$ | $\mathcal{I} \models (A \rightarrow B)$ | $\mathcal{I} \models (\neg B \rightarrow A)$ | $\mathcal{I} \models \varphi$ |
|---|---|---|---|---|---|
| 0 | 0 | Yes | Yes | No | Yes |
| 0 | 1 | No | Yes | Yes | Yes |
| 1 | 0 | Yes | No | Yes | Yes |
| 1 | 1 | No | Yes | Yes | Yes |





# Connection Between Formula Properties and Truth Tables

A propositional formula $\varphi$ is

- satisfiable if $\varphi$ has at least one model
  $\rightsquigarrow$ result in at least one row is "Yes"
- unsatisfiable if $\varphi$ is not satisfiable
  $\rightsquigarrow$ result in all rows is "No"
- valid (or a tautology) if $\varphi$ is true under every interpretation
  $\rightsquigarrow$ result in all rows is "Yes"
- falsifiable if $\varphi$ is no tautology
  $\rightsquigarrow$ result in at least one row is "No"

## Main Disadvantage of Truth Tables

How big is a truth table with $n$ atomic propositions?

| | |
|---|---|
| 1 | 2 interpretations (rows) |
| 2 | 4 interpretations (rows) |
| 3 | 8 interpretations (rows) |
| $n$ | ??? interpretations |

Some examples: $2^{10} = 1024$, $2^{20} = 1048576$, $2^{30} = 1073741824$

$\rightsquigarrow$ not viable for larger formulas; we need a different solution

- more on difficulty of satisfiability etc.: Theory of Computer Science course
- practical algorithms: Foundations of AI course





# D2.2 Equivalences





## Equivalent Formulas

**Definition (Equivalence of Propositional Formulas)**

Two propositional formulas $\varphi$ and $\psi$ over $A$ are (logically) equivalent ($\varphi \equiv \psi$) if for all interpretations $\mathcal{I}$ for $A$ it is true that $\mathcal{I} \models \varphi$ if and only if $\mathcal{I} \models \psi$.

German: logisch äquivalent





## Equivalent Formulas: Example

$$((\varphi \lor \psi) \lor \chi) \equiv (\varphi \lor (\psi \lor \chi))$$

| $\mathcal{I} \models \varphi$ | $\mathcal{I} \models \psi$ | $\mathcal{I} \models \chi$ | $\mathcal{I} \models (\varphi \lor \psi)$ | $\mathcal{I} \models (\psi \lor \chi)$ | $\mathcal{I} \models ((\varphi \lor \psi) \lor \chi)$ | $\mathcal{I} \models (\varphi \lor (\psi \lor \chi))$ |
|---|---|---|---|---|---|---|
| No | No | No | No | No | No | No |
| No | No | Yes | No | Yes | Yes | Yes |
| No | Yes | No | Yes | Yes | Yes | Yes |
| No | Yes | Yes | Yes | Yes | Yes | Yes |
| Yes | No | No | Yes | No | Yes | Yes |
| Yes | No | Yes | Yes | Yes | Yes | Yes |
| Yes | Yes | No | Yes | Yes | Yes | Yes |
| Yes | Yes | Yes | Yes | Yes | Yes | Yes |

## Some Equivalences (1)

$$(\varphi \land \varphi) \equiv \varphi$$
$$(\varphi \lor \varphi) \equiv \varphi \qquad \text{(idempotence)}$$

$$(\varphi \land \psi) \equiv (\psi \land \varphi)$$
$$(\varphi \lor \psi) \equiv (\psi \lor \varphi) \qquad \text{(commutativity)}$$

$$((\varphi \land \psi) \land \chi) \equiv (\varphi \land (\psi \land \chi))$$
$$((\varphi \lor \psi) \lor \chi) \equiv (\varphi \lor (\psi \lor \chi)) \qquad \text{(associativity)}$$

German: Idempotenz, Kommutativität, Assoziativität





## Some Equivalences (2)

$$(\varphi \land (\varphi \lor \psi)) \equiv \varphi$$
$$(\varphi \lor (\varphi \land \psi)) \equiv \varphi \qquad \text{(absorption)}$$

$$(\varphi \land (\psi \lor \chi)) \equiv ((\varphi \land \psi) \lor (\varphi \land \chi))$$
$$(\varphi \lor (\psi \land \chi)) \equiv ((\varphi \lor \psi) \land (\varphi \lor \chi)) \qquad \text{(distributivity)}$$

German: Absorption, Distributivität





## Some Equivalences (3)

$$\neg\neg\varphi \equiv \varphi \qquad \text{(double negation)}$$

$$\neg(\varphi \land \psi) \equiv (\neg\varphi \lor \neg\psi)$$
$$\neg(\varphi \lor \psi) \equiv (\neg\varphi \land \neg\psi) \qquad \text{(De Morgan's rules)}$$

$$(\varphi \lor \psi) \equiv \varphi \text{ if } \varphi \text{ tautology}$$
$$(\varphi \land \psi) \equiv \psi \text{ if } \varphi \text{ tautology} \qquad \text{(tautology rules)}$$

$$(\varphi \lor \psi) \equiv \psi \text{ if } \varphi \text{ unsatisfiable}$$
$$(\varphi \land \psi) \equiv \varphi \text{ if } \varphi \text{ unsatisfiable} \qquad \text{(unsatisfiability rules)}$$

German: Doppelnegation, De Morgansche Regeln, Tautologieregeln, Unerfüllbarkeitsregeln





## Substitution Theorem

**Theorem (Substitution Theorem)**

*Let $\varphi$ and $\varphi'$ be equivalent propositional formulas over A.*
*Let $\psi$ be a propositional formula with (at least) one occurrence of the subformula $\varphi$.*
*Then $\psi$ is equivalent to $\psi'$, where $\psi'$ is constructed from $\psi$ by replacing an occurrence of $\varphi$ in $\psi$ with $\varphi'$.*

German: Ersetzbarkeitstheorem

(without proof)

# Application of Equivalences: Example

$$
\begin{aligned}
(P \land (Q \lor \lnot P)) &\equiv ((P \land Q) \lor (P \land \lnot P)) && \text{(distributivity)} \\
&\equiv ((P \land \lnot P) \lor (P \land Q)) && \text{(commutativity)} \\
&\equiv (P \land Q) && \text{(unsatisfiability rule)}
\end{aligned}
$$