Discrete Mathematics in Computer Science

D1. Syntax and Semantics of Propositional Logic

Malte Helmert, Gabriele Röger

University of Basel

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D1.1 Introduction to Formal Logic

Why Logic?

- formalizing mathematics
  - What is a true statement?
  - What is a valid proof?
  - What can and cannot be proved?
- basis of many tools in computer science
  - design of digital circuits
  - semantics of databases; query optimization
  - meaning of programming languages
  - verification of safety-critical hardware/software
  - knowledge representation in artificial intelligence
  - logic-based programming languages (e.g. Prolog)
  - ...

Application: Logic Programming I

Declarative approach: Describe what to accomplish, not how to accomplish it.

Example (Map Coloring)

Color each region in a map with a limited number of colors so that no two adjacent regions have the same color.

This is a hard problem!

Application: Logic Programming II

Prolog program

```
color(red). color(blue). color(green). color(yellow).

differentColor(ColorA, ColorB) :-
    color(ColorA), color(ColorB),
    ColorA \= ColorB.

switzerland(AG, AI, AR, BE, BL, BS, FR, GE, GL, GR,
            JU, LU, NE, NW, OW, SG, SH, SO, SZ, TG,
            TI, UR, VD, VS, ZG, ZH) :-
    differentColor(AG, BE), differentColor(AG, BL),
    ...
    differentColor(VD, VS), differentColor(ZH, ZG).
```

What Logic is About

General Question:

- Given some knowledge about the world (a knowledge base)
- what can we derive from it?
- And on what basis may we argue?

$\rightsquigarrow$ logic

Goal: "mechanical" proofs

- formal "game with letters"
- detached from a concrete meaning

Running Example

What's the secret of your long life?

I am on a strict diet: If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

Exercise from U. Schöning: Logik für Informatiker
Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

Propositional Logic

Propositional logic is a simple logic without numbers or objects.

Building blocks of propositional logic:

- propositions are statements that can be either true or false
- atomic propositions cannot be split into subpropositions
- logical connectives connect propositions to form new ones

German: Aussagenlogik, Aussage, atomare Aussage, Junktoren/logische Verknüpfungen

Examples for Building Blocks

If I don't drink beer to a meal, then I always eat fish. Whenever I have fish and beer with the same meal, I abstain from ice cream. When I eat ice cream or don't drink beer, then I never touch fish.

- Every sentence is a proposition that consists of subpropositions (e. g., "eat ice cream or don't drink beer").
- atomic propositions "drink beer", "eat fish", "eat ice cream"
- logical connectives "and", "or", negation, "if, then"

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Challenges with Natural Language

If I don't drink beer to a meal, then I always eat fish.
Whenever I have fish and beer with the same meal, I abstain from ice cream.
When I eat ice cream or don't drink beer, then I never touch fish.

- "irrelevant" information

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- "irrelevant" information
- different formulations for the same connective/proposition

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Picture courtesy of graur razvan ionut / FreeDigitalPhotos.net

Challenges with Natural Language

If not DrinkBeer, then EatFish.
If EatFish and DrinkBeer,
then not EatIceCream.
If EatIceCream or not DrinkBeer,
then not EatFish.

- “irrelevant” information
- different formulations for the same connective/proposition

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What is Next?

- What are meaningful (well-defined) sequences of atomic propositions and connectives?
  "if then EatIceCream not or DrinkBeer and" not meaningful
  $\rightarrow$ syntax
- What does it mean if we say that a statement is true?
  Is "DrinkBeer and EatFish" true?
  $\rightarrow$ semantics
- When does a statement logically follow from another?
  Does "EatFish" follow from "if DrinkBeer, then EatFish"?
  $\rightarrow$ logical entailment

German: Syntax, Semantik, logische Folgerung

D1.2 Syntax of Propositional Logic

Syntax of Propositional Logic

Definition (Syntax of Propositional Logic)

Let A be a set of atomic propositions. The set of propositional formulas (over A) is inductively defined as follows:

- Every atom $a \in A$ is a propositional formula over A.
- If φ is a propositional formula over A, then so is its negation $\neg\varphi$.
- If φ and ψ are propositional formulas over A, then so is the conjunction $(\varphi \land \psi)$.
- If φ and ψ are propositional formulas over A, then so is the disjunction $(\varphi \lor \psi)$.

The implication $(\varphi \rightarrow \psi)$ is an abbreviation for $(\neg\varphi \lor \psi)$.
The biconditional $(\varphi \leftrightarrow \psi)$ is an abbrev. for $((\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi))$.
German: atomare Aussage, aussagenlogische Formel, Atom, Negation, Konjunktion, Disjunktion, Implikation, Bikonditional

Syntax: Examples

Which of the following sequences of symbols are propositional formulas over the set of all possible letter sequences? Which kinds of formula are they (atom, conjunction, ...)?

- $(\mathsf{A} \land (\mathsf{B} \lor \mathsf{C}))$
- $\neg(\land \mathsf{Rain} \lor \mathsf{StreetWet})$
- $\neg(\mathsf{Rain} \lor \mathsf{StreetWet})$
- $((\mathsf{EatFish} \land \mathsf{DrinkBeer}) \rightarrow \neg\mathsf{EatIceCream})$
- $\mathsf{Rain} \land \neg\mathsf{Rain}$
- $\neg(\mathsf{A} = \mathsf{B})$
- $(\mathsf{A} \land \neg(\mathsf{B} \leftrightarrow)\mathsf{C})$
- $((\mathsf{A} \leq \mathsf{B}) \land \mathsf{C})$
- $(\mathsf{A} \lor \neg(\mathsf{B} \leftrightarrow \mathsf{C}))$
- $((\mathsf{A}_1 \land \mathsf{A}_2) \lor \neg(\mathsf{A}_3 \leftrightarrow \mathsf{A}_2))$

D1.3 Semantics of Propositional Logic

Meaning of Propositional Formulas?

So far propositional formulas are only symbol sequences without any meaning.

For example, what does this mean:
$((\text{EatFish} \land \text{DrinkBeer}) \rightarrow \neg\text{EatIceCream})$?

▷ We need semantics!

Semantics of Propositional Logic

Definition (Semantics of Propositional Logic)

A truth assignment (or interpretation) for a set of atomic propositions A is a function $\mathcal{I} : A \to \{0, 1\}$.

A propositional formula φ (over A) holds under $\mathcal{I}$ (written as $\mathcal{I} \models \varphi$) according to the following definition:

$$
\begin{array}{lll}
\mathcal{I} \models a & \text{iff} & \mathcal{I}(a) = 1 \qquad\qquad (\text{for } a \in A) \\
\mathcal{I} \models \neg\varphi & \text{iff} & \text{not } \mathcal{I} \models \varphi \\
\mathcal{I} \models (\varphi \land \psi) & \text{iff} & \mathcal{I} \models \varphi \text{ and } \mathcal{I} \models \psi \\
\mathcal{I} \models (\varphi \lor \psi) & \text{iff} & \mathcal{I} \models \varphi \text{ or } \mathcal{I} \models \psi
\end{array}
$$

Question: should we define semantics of $(\varphi \to \psi)$ and $(\varphi \leftrightarrow \psi)$?

German: Wahrheitsbelegung/Interpretation, φ gilt unter $\mathcal{I}$

Semantics of Propositional Logic: Terminology

- For $\mathcal{I} \models \varphi$ we also say $\mathcal{I}$ is a model of φ and that φ is true under $\mathcal{I}$.
- If φ does not hold under $\mathcal{I}$, we write this as $\mathcal{I} \not\models \varphi$ and say that $\mathcal{I}$ is no model of φ and that φ is false under $\mathcal{I}$.
- Note: $\models$ is not part of the formula but part of the meta language (speaking about a formula).

German: $\mathcal{I}$ ist ein/kein Modell von φ; φ ist wahr/falsch unter $\mathcal{I}$; Metasprache

Exercise

Consider the set $A = \{\mathsf{X}, \mathsf{Y}, \mathsf{Z}\}$ of atomic propositions and formula $\varphi = (\mathsf{X} \land \neg\mathsf{Y})$.

Specify an interpretation $\mathcal{I}$ for A with $\mathcal{I} \models \varphi$.

Semantics: Example (1)

$$A = \{\mathsf{DrinkBeer}, \mathsf{EatFish}, \mathsf{EatIceCream}\}$$
$$\mathcal{I} = \{\mathsf{DrinkBeer} \mapsto 1, \mathsf{EatFish} \mapsto 0, \mathsf{EatIceCream} \mapsto 1\}$$
$$\varphi = (\neg\mathsf{DrinkBeer} \rightarrow \mathsf{EatFish})$$

Do we have $\mathcal{I} \models \varphi$?

Semantics: Example (2)

Goal: prove $\mathcal{I} \models \varphi$.

Let us use the definitions we have seen:

$$\begin{aligned}\mathcal{I} \models \varphi &\text{ iff } \mathcal{I} \models (\neg\mathsf{DrinkBeer} \rightarrow \mathsf{EatFish}) \\ &\text{ iff } \mathcal{I} \models (\neg\neg\mathsf{DrinkBeer} \lor \mathsf{EatFish}) \\ &\text{ iff } \mathcal{I} \models \neg\neg\mathsf{DrinkBeer} \text{ or } \mathcal{I} \models \mathsf{EatFish}\end{aligned}$$

This means that if we want to prove $\mathcal{I} \models \varphi$, it is sufficient to prove

$$\mathcal{I} \models \neg\neg\mathsf{DrinkBeer}$$

or to prove

$$\mathcal{I} \models \mathsf{EatFish}.$$

We attempt to prove the first of these statements.

Semantics: Example (3)

New goal: prove $\mathcal{I} \models \neg\neg\mathsf{DrinkBeer}$.

We again use the definitions:

$$\begin{aligned}\mathcal{I} \models \neg\neg\mathsf{DrinkBeer} &\text{ iff not } \mathcal{I} \models \neg\mathsf{DrinkBeer}\\ &\text{ iff not not } \mathcal{I} \models \mathsf{DrinkBeer}\\ &\text{ iff } \mathcal{I} \models \mathsf{DrinkBeer}\\ &\text{ iff } \mathcal{I}(\mathsf{DrinkBeer}) = 1\end{aligned}$$

The last statement is true for our interpretation $\mathcal{I}$.

To write this up as a proof of $\mathcal{I} \models \varphi$,
we can go through this line of reasoning back-to-front,
starting from our assumptions and ending with the conclusion
we want to show.

Semantics: Example (4)

Let $\mathcal{I} = \{\mathsf{DrinkBeer} \mapsto 1, \mathsf{EatFish} \mapsto 0, \mathsf{EatIceCream} \mapsto 1\}$.

Proof that $\mathcal{I} \models (\neg\mathsf{DrinkBeer} \rightarrow \mathsf{EatFish})$:

1. We have $\mathcal{I} \models \mathsf{DrinkBeer}$
   (uses defn. of $\models$ for atomic props. and fact $\mathcal{I}(\mathsf{DrinkBeer}) = 1$).
2. From (1), we get $\mathcal{I} \not\models \neg\mathsf{DrinkBeer}$
   (uses defn. of $\models$ for negations).
3. From (2), we get $\mathcal{I} \models \neg\neg\mathsf{DrinkBeer}$
   (uses defn. of $\models$ for negations).
4. From (3), we get $\mathcal{I} \models (\neg\neg\mathsf{DrinkBeer} \vee \psi)$ for all formulas ψ,
   in particular $\mathcal{I} \models (\neg\neg\mathsf{DrinkBeer} \vee \mathsf{EatFish})$
   (uses defn. of $\models$ for disjunctions).
5. From (4), we get $\mathcal{I} \models (\neg\mathsf{DrinkBeer} \rightarrow \mathsf{EatFish})$
   (uses defn. of "$\rightarrow$"). $\square$

Summary

- propositional logic based on atomic propositions
- syntax defines what well-formed formulas are
- semantics defines when a formula is true
- interpretations are the basis of semantics