Discrete Mathematics in Computer Science

C3. Acyclicity

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Acyclic (Di-) Graphs

Acyclic

Similarly to connectedness, the presence or absence of cycles is an important practical property for (di-) graphs.

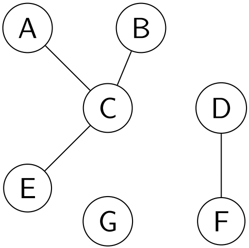
Definition (acyclic, forest, DAG)

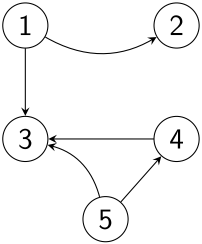
A graph or digraph G is called acyclic if there exists no cycle in G.

An acyclic graph is also called a forest.
An acyclic digraph is also called a DAG (directed acyclic graph).

German: azyklisch/kreisfrei, Wald, DAG

Acyclic (Di-) Graphs – Example





Trees

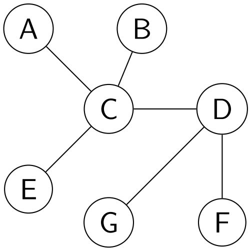
Definition (tree)

A connected forest is called a tree.

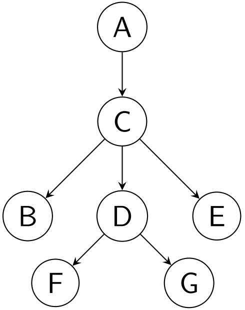
German: Baum

- Tree is also a word for a recursive data structure, which consists of either a leaf or a parent node with one or more children, which are themselves trees.
- This other kind of tree is also called a rooted tree to distinguish it from a tree as a graph.
- The two meanings of "tree" are distinct but closely related.

Tree Graphs vs. Rooted Trees – Example (1)

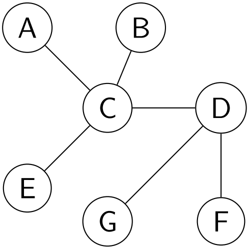


tree graph

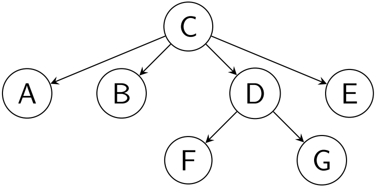


rooted tree with root A

Tree Graphs vs. Rooted Trees – Example (2)

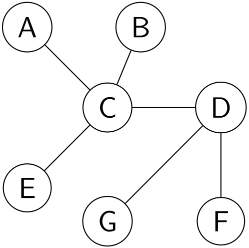


tree graph

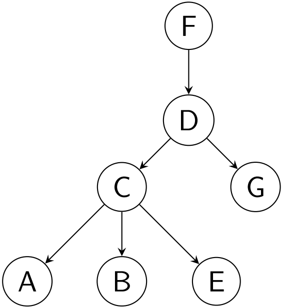


rooted tree with root C

Tree Graphs vs. Rooted Trees – Example (3)



tree graph



rooted tree with root F

From Tree Graphs to Rooted Trees

General procedure for converting tree graphs into rooted trees:

- Select any vertex v. Make v the root of the tree.
- Initially, v is the only pending vertex,
  and there are no processed vertices.
- As long as there are pending vertices:
  - Select any pending vertex u.
  - Make all neighbours v of u that are not yet processed children of u and mark them as pending.
  - Change u from pending to processed.

We do not prove that this procedure always works. A proof of correctness can be given based on the results we show next.

Unique Paths in Trees

Unique Paths in Trees

Theorem

Let $G = (V, E)$ be a graph.
Then G is a tree iff there exists exactly one path
from any vertex $u \in V$ to any vertex $v \in V$.

Unique Paths In Trees – Proof (1)

Proof.

(⇒): G is a tree. Let $u, v \in V$.
We must show that there exists exactly one path from u to v.

We know that at least one path exists because G is connected.

It remains to show that there cannot be two paths from u to v.
If $u = v$, there is only one path (the empty one).
(Any longer path would have to repeat a vertex.)

We assume that there exist two different paths from u to v ($u \neq v$) and derive a contradiction. . . .

Unique Paths In Trees – Proof (2)

Proof (continued).

Let $\pi = \langle v_0, v_1, \ldots, v_n \rangle$ and $\pi' = \langle v'_0, v'_1, \ldots, v'_m \rangle$ be the two paths (with $v_0 = v'_0 = u$ and $v_n = v'_m = v$).
Let i be the smallest index with $v_i \neq v'_i$, which must exist because the two paths are different, and neither can be a prefix of the other (else v would be repeated in the longer path).
We have $i \geq 1$ because $v_0 = v'_0$.
Let $j \geq i$ be the smallest index such that $v_j = v'_k$ for some $k \geq i$.
Such an index must exist because $v_n = v'_m$.
Then $\langle v_{i-1}, \ldots, v_{j-1}, v'_k, \ldots, v'_{i-1} \rangle$ is a cycle,
which contradicts the requirement that G is a tree. ...

Unique Paths In Trees – Proof (3)

Proof (continued).

(⇐): For all $u, v \in V$, there exists exactly one path from u to v.
We must show that G is a tree, i.e., is connected and acyclic.

Because there exist paths from all u to all v, G is connected.

Proof by contradiction: assume that there exists a cycle in G,
$\pi = \langle u, v_1, \dots, v_n, u \rangle$ with $n \geq 2$.
(Note that all cycles have length at least 3.)
From the definition of cycles, we have $v_1 \neq v_n$.

Then $\langle u, v_1 \rangle$ and $\langle u, v_n, \dots, v_1 \rangle$ are two different paths from u to v_1, contradicting that there exists exactly one path from every vertex to every vertex. Hence G must be acyclic. □

Leaves and Edge Counts in Trees and Forests

Leaves in Trees

Definition

Let $G = (V, E)$ be a tree.
A leaf of G is a vertex $v \in V$ with $\deg(v) \leq 1$.

Note: The case $\deg(v) = 0$ only occurs in single-vertex trees ($|V| = 1$). In trees with at least two vertices, vertices with degree 0 cannot exist because this would make the graph unconnected.

Theorem

Let $G = (V, E)$ be a tree with $|V| \geq 2$.
Then G has at least two leaves.

Leaves in Trees – Proof

Proof.

Let $\pi = \langle v_0, \dots, v_n \rangle$ be path in G with maximal length among all paths in G.
Because $|V| \geq 2$, we have $n \geq 1$ (else G would not be connected).

We show that vertex v_n has degree 1: v_{n-1} is a neighbour in G.
Assume that it were not the only neighbour of v_n in G, so u is another neighbour of v_n. Then:

- If u is not on the path, then $\langle v_0, \dots, v_n, u \rangle$ is a longer path: contradiction.
- If u is on the path, then $u = v_i$ for some $i \neq n$ and $i \neq n - 1$. Then $\langle v_i, \dots, v_n, v_i \rangle$ is a cycle: contradiction.

By reversing π we can show $\deg(v_0) = 1$ in the same way. □

Edges in Trees

Theorem

Let $G = (V, E)$ be a tree with $V \neq \emptyset$.
Then $|E| = |V| - 1$.

Edges in Trees – Proof (1)

Proof.

Proof by induction over $n = |V|$.

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Induction base ($n = 1$):
Then G has 1 vertex and 0 edges.
We get $|E| = 0 = 1 - 1 = |V| - 1$.

Edges in Trees – Proof (1)

Proof.

Proof by induction over $n = |V|$.

Induction base ($n = 1$):
Then G has 1 vertex and 0 edges.
We get $|E| = 0 = 1 - 1 = |V| - 1$.

Induction step ($n \rightarrow n + 1$):
Let $G = (V, E)$ be a tree with $n + 1$ vertices ($n \geq 1$).
From the previous result, G has a leaf u.
Let v be the only neighbour of u.
Let $e = \{u, v\}$ be the connecting edge. ...

Edges in Trees – Proof (2)

Proof (continued).

Consider the graph $G' = (V', E')$
with $V' = V \setminus \{u\}$ and $E' = E \setminus \{e\}$.

- G' is acyclic: every cycle in G' would also be present in G (contradiction).
- G' is connected: for all vertices $w \neq u$ and $w' \neq u$,
  G has a path π from w to w' because G is connected.
  Path π cannot include u because u has only one neighbour, so traversing u requires repeating v. Hence π is also a path in G'.

Hence G' is a tree with n vertices, and we can apply
the induction hypothesis, which gives $|E'| = |V'| - 1$.
It follows that
$|E| = |E'| + 1 = (|V'| - 1) + 1 = (|V'| + 1) - 1 = |V| - 1$. □

Edges in Forests

Theorem

Let $G = (V, E)$ be a forest.
Let C be the set of connected components of G.
Then $|E| = |V| - |C|$.

This result generalizes the previous one.

Edges in Forests – Proof

Proof.

Let $C = \{C_1, \ldots, C_k\}$.

For $1 \leq i \leq k$, let $G_i = (C_i, E_i)$ be G restricted to C_i, i.e., the graph whose vertices are C_i and whose edges are the edges $e \in E$ with $e \subseteq C_i$.

We have $|V| = \sum_{i=1}^{k} |C_i|$ because the connected components form a partition of V.

We have $|E| = \sum_{i=1}^{k} |E_i|$ because every edge belongs to exactly one connected component. (Note that there cannot be edges between different connected components.)

Every graph G_i is a tree with at least one vertex: it is connected because its vertices form a connected component, and it is acyclic because G is. This implies $|E_i| = |C_i| - 1$.

Putting this together, we get $|E| = \sum_{i=1}^{k} |E_i| = \sum_{i=1}^{k}(|C_i| - 1) = \sum_{i=1}^{k} |C_i| - k = |V| - |C|$. □

Characterizations of Trees

Characterizations of Trees

Theorem

Let $G = (V, E)$ be a graph with $V \neq \emptyset$.
The following statements are equivalent:

1. *G is a tree.*
2. *G is acyclic and connected.*
3. *G is acyclic and $|E| = |V| - 1$.*
4. *G is connected and $|E| = |V| - 1$.*
5. *For all $u, v \in V$ there exists exactly one path from u to v.*

Characterizations of Trees – Proof (1)

Reminder:

(1) G is a tree.
(2) G is acyclic and connected.
(3) G is acyclic and $|E| = |V| - 1$.
(4) G is connected and $|E| = |V| - 1$.

(5) For all $u, v \in V$ there exists exactly one path from u to v.

Proof.

We know already:

- (1) and (2) are equivalent by definition of trees.
- We have shown that (1) and (5) are equivalent.
- We have shown that (1) implies (3) and (4).

We complete the proof by showing (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (2). ...

Characterizations of Trees – Proof (2)

Reminder:

(2) G is acyclic and connected.

(3) G is acyclic and $|E| = |V| - 1$.

Proof (continued).

(3) $\Rightarrow$ (2):

Because G is acyclic, it is a forest.

From the previous result, we have $|E| = |V| - |C|$,

where C are the connected components of G.

Characterizations of Trees – Proof (2)

Reminder:

(2) G is acyclic and connected.

(3) G is acyclic and $|E| = |V| - 1$.

Proof (continued).

(3) $\Rightarrow$ (2):

Because G is acyclic, it is a forest.

From the previous result, we have $|E| = |V| - |C|$,

where C are the connected components of G.

But we also know $|E| = |V| - 1$. This implies $|C| = 1$.

Hence G is connected and therefore a tree. . . .

Characterizations of Trees – Proof (3)

Reminder:

(2) G is acyclic and connected.

(4) G is connected and $|E| = |V| - 1$.

Proof (continued).

(4) $\Rightarrow$ (2):

In graphs that are not acyclic, we can remove an edge without changing the connected components: if $\langle v_0, \ldots, v_n, v_0 \rangle$ ($n \geq 2$) is a cycle, remove the edge $\{v_0, v_1\}$ from the graph.

Every walk using this edge can substitute $\langle v_1, \ldots, v_n, v_0 \rangle$ (or the reverse path) for it.

Characterizations of Trees – Proof (3)

Reminder:

(2) G is acyclic and connected.

(4) G is connected and $|E| = |V| - 1$.

Proof (continued).

(4) $\Rightarrow$ (2):

In graphs that are not acyclic, we can remove an edge without changing the connected components: if $\langle v_0, \dots, v_n, v_0 \rangle$ ($n \geq 2$) is a cycle, remove the edge $\{v_0, v_1\}$ from the graph.
Every walk using this edge can substitute $\langle v_1, \dots, v_n, v_0 \rangle$ (or the reverse path) for it.

Iteratively remove edges from G in this way while preserving connectedness until this is no longer possible. The resulting graph (V, E') is acyclic and connected and therefore a tree.

Characterizations of Trees – Proof (3)

Reminder:

(2) G is acyclic and connected.

(4) G is connected and $|E| = |V| - 1$.

Proof (continued).

(4) $\Rightarrow$ (2):

In graphs that are not acyclic, we can remove an edge without changing the connected components: if $\langle v_0, \dots, v_n, v_0 \rangle$ $(n \geq 2)$ is a cycle, remove the edge $\{v_0, v_1\}$ from the graph.
Every walk using this edge can substitute $\langle v_1, \dots, v_n, v_0 \rangle$ (or the reverse path) for it.

Iteratively remove edges from G in this way while preserving connectedness until this is no longer possible. The resulting graph (V, E') is acyclic and connected and therefore a tree.

This implies $|E'| = |V| - 1$, but we also have $|E| = |V| - 1$.
This yields $|E| = |E'|$ and hence $E' = E$: the number of edges removable in this way must be 0. Hence G is already acyclic. □