

Discrete Mathematics in Computer Science

C2. Paths and Connectivity

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November 11, 2024

Walks, Paths, Tours and Cycles

Traversing Graphs

- When dealing with graphs, we are often not just interested in the neighbours, but also in the **neighbours of neighbours**, the **neighbours of neighbours of neighbours**, etc.
- Similarly, for digraphs we often want to follow longer chains of successors (or chains of predecessors).

Traversing Graphs

- When dealing with graphs, we are often not just interested in the neighbours, but also in the **neighbours of neighbours**, the **neighbours of neighbours of neighbours**, etc.
- Similarly, for digraphs we often want to follow longer chains of successors (or chains of predecessors).

Examples:

- circuits: follow predecessors of signals to identify possible causes of faulty signals
- pathfinding: follow edges/arcs to find paths
- control flow graphs: follow arcs to identify dead code
- computer networks: determine if part of the network is unreachable

Walks

Definition (Walk)

A **walk** of **length** n in a graph (V, E) is a tuple $\langle v_0, v_1, \dots, v_n \rangle \in V^{n+1}$ s.t. $\{v_i, v_{i+1}\} \in E$ for all $0 \leq i < n$.

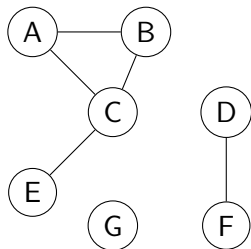
A **walk** of **length** n in a digraph (N, A) is a tuple $\langle v_0, v_1, \dots, v_n \rangle \in N^{n+1}$ s.t. $(v_i, v_{i+1}) \in A$ for all $0 \leq i < n$.

German: Wanderung

Notes:

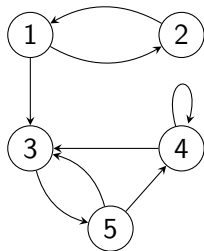
- The length of the walk does not equal the length of the tuple!
- The case $n = 0$ is allowed.
- Vertices may repeat along a walk.

Walks – Example



examples of walks:

- $\langle B, C, A \rangle$
- $\langle B, C, A, B \rangle$
- $\langle D, F, D \rangle$
- $\langle B, A, B, C, E \rangle$
- $\langle B \rangle$



examples of walks:

- $\langle 4, 4, 4, 4 \rangle$
- $\langle 3, 5, 3, 5 \rangle$
- $\langle 2, 1, 3 \rangle$
- $\langle 4 \rangle$
- $\langle 4, 4 \rangle$

Walks – Terminology

Definition

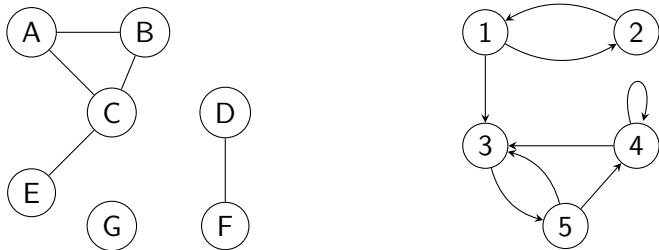
Let $\pi = \langle v_0, \dots, v_n \rangle$ be a walk in a graph or digraph G .

- We say π is a walk **from** v_0 **to** v_n .
- A walk with $v_i \neq v_j$ for all $0 \leq i < j \leq n$ is called a **path**.
- A walk of length 0 is called an **empty** walk/path.
- A walk with $v_0 = v_n$ is called a **tour**.
- A tour with $n \geq 1$ (digraphs) or $n \geq 3$ (graphs) and $v_i \neq v_j$ for all $1 \leq i < j \leq n$ is called a **cycle**.

German: von/nach, Pfad, leer, Tour, Zyklus

Note: Terminology is not very consistent in the literature.

Walks, Paths, Tours, Cycles – Example



Which walks are paths, tours, cycles?

- $\langle B, C, A \rangle$
- $\langle B, C, A, B \rangle$
- $\langle D, F, D \rangle$
- $\langle B, A, B, C, E \rangle$
- $\langle B \rangle$
- $\langle 4, 4, 4, 4 \rangle$
- $\langle 3, 5, 3, 5 \rangle$
- $\langle 2, 1, 3 \rangle$
- $\langle 4 \rangle$
- $\langle 4, 4 \rangle$

Reachability

Reachability

Definition (successor and reachability)

Let G be a graph (digraph).

The **successor relation** S_G and **reachability relation** R_G are relations over the vertices/nodes of G defined as follows:

- $(u, v) \in S_G$ iff $\{u, v\}$ is an edge ((u, v) is an arc) of G
- $(u, v) \in R_G$ iff there exists a walk from u to v

If $(u, v) \in R_G$, we say that **v is reachable from u** .

German: Nachfolger-/Erreichbarkeitsrelation, erreichbar

Reachability as Closure

Recall the n -fold composition R_n of a relation R over set S (Chapter B4):

- $R_0 = \{(x, x) \mid x \in S\}$
- $R_n = R \circ R_{n-1}$ for $n \geq 1$

Theorem

*Let G be a graph or digraph. Then:
 $(u, v) \in S_G^n$ iff there exists a walk of length n from u to v .*

Corollary

Let G be a graph or digraph. Then $R_G = \bigcup_{n=0}^{\infty} (S_G)_n$.

In other words, the reachability relation is the reflexive transitive closure of the successor relation.

Reachability as Closure – Proof (1)

Proof.

To simplify notation, we assume $G = (N, A)$ is a digraph.

Graphs are analogous.

Proof by induction over n .

...

Reachability as Closure – Proof (1)

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Graphs are analogous.

Proof by induction over n .

induction base ($n = 0$):

By definition of the 0-fold composition, we have $(u, v) \in (S_G)_0$ iff $u = v$, and a walk of length 0 from u to v exists iff $u = v$.

Hence, the two conditions are equivalent.

...

Reachability as Closure – Proof (2)

Proof (continued).

induction step ($n \rightarrow n + 1$):



Reachability as Closure – Proof (2)

Proof (continued).

induction step ($n \rightarrow n + 1$):

(\Rightarrow) : Let $(u, v) \in (S_G)_{n+1}$.

By definition of R_{n+1} , we get $(u, v) \in S_G \circ (S_G)_n$.

By definition of \circ there exists w with $(u, w) \in (S_G)_n$ and $(w, v) \in S_G$.

From the induction hypothesis, there exists a length- n walk $\langle x_0, \dots, x_n \rangle$ with $x_0 = u$ and $x_n = w$.

Then $\langle x_0, \dots, x_n, v \rangle$ is a length- $(n + 1)$ walk from u to v .



Reachability as Closure – Proof (2)

Proof (continued).

induction step ($n \rightarrow n + 1$):

(\Rightarrow) : Let $(u, v) \in (S_G)_{n+1}$.

By definition of R_{n+1} , we get $(u, v) \in S_G \circ (S_G)_n$.

By definition of \circ there exists w with $(u, w) \in (S_G)_n$ and $(w, v) \in S_G$.

From the induction hypothesis, there exists a length- n walk $\langle x_0, \dots, x_n \rangle$ with $x_0 = u$ and $x_n = w$.

Then $\langle x_0, \dots, x_n, v \rangle$ is a length- $(n + 1)$ walk from u to v .

(\Leftarrow) : Let $\langle x_0, \dots, x_{n+1} \rangle$ be a length- $(n + 1)$ walk from u to v ($x_0 = u, x_{n+1} = v$). Then $(x_n, x_{n+1}) = (x_n, v) \in A$.

Also, $\langle x_0, \dots, x_n \rangle$ is a length- n walk from x_0 to x_n .

From the IH we get $(u, x_n) = (x_0, x_n) \in (S_G)_n$.

Together with $(x_n, v) \in S_G$ this shows

$(u, v) \in S_G \circ (S_G)_n = (S_G)_{n+1}$. □

Connected Components

Overview

- In this section, we study reachability of graphs in more depth.
- We show that it makes no difference whether we define reachability in terms of walks or paths, and that reachability in graphs is an **equivalence relation**.
- This leads to the **connected components** of a graph.
- In digraphs, reachability is not always an equivalence relation.
- However, we can define two variants of reachability that give rise to **weakly** or **strongly connected components**.

Walks vs. Paths

Theorem

Let G be a graph or digraph.

There exists a path from u to v iff there exists a walk from u to v .

In other words, there is a path from u to v iff v is reachable from u .



Walks vs. Paths

Theorem

Let G be a graph or digraph.

There exists a path from u to v iff there exists a walk from u to v .

In other words, there is a path from u to v iff v is reachable from u .

Proof.

(\Rightarrow): obvious because paths are special cases of walks

(\Leftarrow): Proof by contradiction. Assume there exist u, v such that there exists a walk from u to v , but no path. Let $\pi = \langle w_0, \dots, w_n \rangle$ be such a counterexample walk of minimal length.

Because π is not a path, some vertex/node must repeat.

Select i and j with $i < j$ and $w_i = w_j$.

Then $\pi' = \langle w_0, \dots, w_i, w_{j+1}, \dots, w_n \rangle$ also is a walk from u to v .

If π' is a path, we have a contradiction.

If not, it is a shorter counterexample: also a contradiction. □

Reachability in Graphs is an Equivalence Relation

Theorem

For every *graph* G , the reachability relation R_G is an *equivalence relation*.

In *directed graphs*, this result does not hold (easy to see).

Proof.

We already know reachability is reflexive and transitive.

To prove symmetry:

$$(u, v) \in R_G$$

\Rightarrow there is a walk $\langle w_0, \dots, w_n \rangle$ from u to v

$\Rightarrow \langle w_n, \dots, w_0 \rangle$ is a walk from v to u

$$\Rightarrow (v, u) \in R_G$$



Connected Components

Definition (connected components, connected)

In a graph G , the equivalence classes of the reachability relation of G are called the **connected components** of G .

A graph is called **connected** if it has at most 1 connected component.

German: Zusammenhangskomponenten, zusammenhängend

Remark: The graph (\emptyset, \emptyset) has 0 connected components. It is the only such graph.

Weakly Connected Components

Definition (weakly connected components, weakly connected)

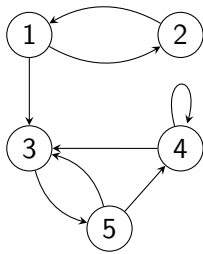
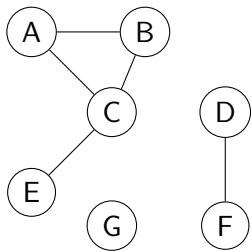
In a digraph G , the equivalence classes of the reachability relation of the induced graph of G are called the **weakly connected components** of G .

A digraph is called **weakly connected** if it has at most 1 weakly connected component.

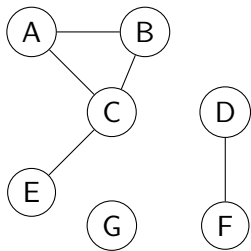
German: schwache Zshk., schwach zusammenhängend

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(Weakly) Connected Components – Example

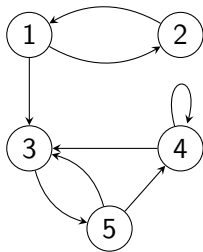


(Weakly) Connected Components – Example



connected components:

- {A, B, C, E}
- {D, F}
- {G}



weakly connected components:

- {1, 2, 3, 4, 5}

Mutual Reachability

Definition (mutually reachable)

Let G be a graph or digraph.

Vertices/nodes u and v in G are called **mutually reachable** if v is reachable from u and u is reachable from v .

We write M_G for the **mutual reachability** relation of G

German: gegenseitig erreichbar

Note: In graphs, $M_G = R_G$. (Why?)

Mutual Reachability is an Equivalence Relation

Theorem

For every *digraph* G , the mutual reachability relation M_G is an *equivalence relation*.

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Proof.

Note that $(u, v) \in M_G$ iff $(u, v) \in R_G$ and $(v, u) \in R_G$.

- **reflexivity:** for all v , we have $(v, v) \in M_G$ because $(v, v) \in R_G$



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- **reflexivity:** for all v , we have $(v, v) \in M_G$ because $(v, v) \in R_G$
- **symmetry:** Let $(u, v) \in M_G$. Then $(v, u) \in M_G$ is obvious.



Mutual Reachability is an Equivalence Relation

Theorem

For every *digraph* G , the mutual reachability relation M_G is an *equivalence relation*.

Proof.

Note that $(u, v) \in M_G$ iff $(u, v) \in R_G$ and $(v, u) \in R_G$.

- **reflexivity:** for all v , we have $(v, v) \in M_G$ because $(v, v) \in R_G$
- **symmetry:** Let $(u, v) \in M_G$. Then $(v, u) \in M_G$ is obvious.
- **transitivity:** Let $(u, v) \in M_G$ and $(v, w) \in M_G$.
Then: $(u, v) \in R_G$, $(v, u) \in R_G$, $(v, w) \in R_G$, $(w, v) \in R_G$.
Transitivity of R_G yields $(u, w) \in R_G$ and $(w, u) \in R_G$,
and hence $(u, w) \in M_G$.



Strongly Connected Components

Definition (strongly connected components, strongly connected)

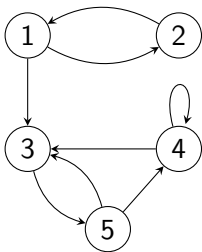
In a digraph G , the equivalence classes of the mutual reachability relation are called the **strongly connected components** of G .

A digraph is called **strongly connected** if it has at most 1 strongly connected component.

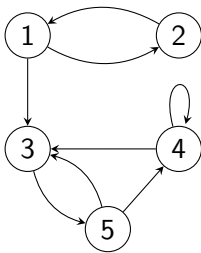
German: starke Zshk., stark zusammenhängend

Remark: The digraph (\emptyset, \emptyset) has 0 strongly connected components. It is the only such digraph.

Strongly Connected Components – Example



Strongly Connected Components – Example



strongly connected components:

- {1, 2}
- {3, 4, 5}