

Discrete Mathematics in Computer Science

B9. Divisibility & Modular Arithmetic

Malte Helmert, Gabriele Röger

University of Basel

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B9.1 Divisibility

B9.2 Modular Arithmetic

B9.1 Divisibility

Divisibility



- ▶ Can we equally share n muffins among m persons without cutting a muffin?
- ▶ If yes then n is a multiple of m and m divides n .
- ▶ We consider a generalization of this concept to the integers.

Divisibility

Definition (divisor, multiple)

Let $m, n \in \mathbb{Z}$. If there exists a $k \in \mathbb{Z}$ such that $mk = n$, we say that m divides n , m is a divisor of n or n is a multiple of m and write this as $m \mid n$.

Which of the following are true?

- ▶ $2 \mid 4$
- ▶ $-2 \mid 4$
- ▶ $2 \mid -4$
- ▶ $4 \mid 2$
- ▶ $3 \mid 4$
- ▶ Every integer divides 0.

German: teilt, Teiler, Vielfaches

Divisibility and Linear Combinations

Theorem (Linear combinations)

Let a, b and d be integers. If $d \mid a$ and $d \mid b$ then for all integers x and y it holds that $d \mid xa + yb$.

Proof.

If $d \mid a$ and $d \mid b$ then there are $k, k' \in \mathbb{Z}$ such that $kd = a$ and $k'd = b$.

It holds for all $x, y \in \mathbb{Z}$ that $xa + yb = xkd + yk'd = (xk + yk')d$. As x, y, k, k' are integers, $xk + yk'$ is integer, thus $d \mid xa + yb$. \square

Some consequences:

- ▶ $d \mid a - b$ iff $d \mid b - a$
- ▶ If $d \mid a$ and $d \mid b$ then $d \mid a + b$ and $d \mid a - b$.
- ▶ If $d \mid a$ then $d \mid -8a$.

Multiplication and Exponentiation

Theorem

Let $a, b, c \in \mathbb{Z}$ and $n \in \mathbb{N}_{>0}$.
If $a \mid b$ then $ac \mid bc$ and $a^n \mid b^n$.

Proof.

If $a \mid b$ there is a $k \in \mathbb{Z}$ such that $ak = b$.

Multiplying both sides with c , we get $cak = cb$ and thus $ca \mid cb$.

From $ak = b$, we also get $b^n = (ak)^n = a^n k^n$, so $a^n \mid b^n$. \square

Partial Order

If we consider only the natural numbers, divisibility is a partial order:

Theorem

Divisibility | over \mathbb{N}_0 is a partial order.

Proof.

- ▶ **reflexivity:** For all $m \in \mathbb{N}_0$ it holds that $m \cdot 1 = m$, so $m \mid m$.
- ▶ **transitivity:** If $m \mid n$ and $n \mid o$ there are $k, k' \in \mathbb{Z}$ such that $mk = n$ and $nk' = o$.
It holds that $o = nk' = mkk'$ and kk' is an integer, so we conclude $m \mid o$.

...

Partial Order

Proof (continued).

► **antisymmetry:** We show that if $m \mid n$ and $n \mid m$ then $m = n$.

If $m = n = 0$, there is nothing to show.

Otherwise, at least one of m and n is positive.

Let this w.l.o.g. (without loss of generality) be m .

If $m \mid n$ and $n \mid m$ then there are $k, k' \in \mathbb{Z}$
such that $mk = n$ and $nk' = m$.

Combining these, we get $m = nk' = mkk'$, which implies
(with $m \neq 0$) that $kk' = 1$.

Since k and k' are integers, this implies $k = k' = 1$ or
 $k = k' = -1$. As $mk = n$, m is positive and n is non-negative,
we can conclude that $k = 1$ and $m = n$.

□

B9.2 Modular Arithmetic

Halloween



- You have m sweets.
- There are k kids showing up for trick-or-treating.
- To keep everything fair, every kid gets the same amount of treats.
- You may enjoy the rest. :-)
- How much does every kid get, how much do you get?

Euclid's Division Lemma

Theorem (Euclid's division lemma)

For all integers a and b with $b \neq 0$
there are **unique** integers q and r
with $a = qb + r$ and $0 \leq r < |b|$.

Number a is called the **dividend**, b the **divisor**, q is the **quotient**
and r the **remainder**.

Without proof.

Examples:

- $a = 18, b = 5$
- $a = 5, b = 18$
- $a = -18, b = 5$
- $a = 18, b = -5$

German: Division mit Rest, Dividend, Divisor, Ganzzahlquotient, Rest

Modulo Operation

- ▶ With $a \bmod b$ we refer to the remainder of Euclidean division.
- ▶ Most programming languages have a built-in operator to compute $a \bmod b$ (for positive integers):

```
int mod = 34 % 7;
// result 6 because 4 * 7 + 6 = 34
```

- ▶ **Common application:** Determine whether a natural number n is even.

```
n % 2 == 0
```

- ▶ Languages behave differently with negative operands!

Halloween



```
def share_sweets(no_kids, no_sweets):
    print("Each kid gets",
          no_sweets // no_kids,
          "of the sweets.")
    print("You may keep",
          no_sweets % no_kids,
          "of the sweets.")
```

Congruence Modulo n

- ▶ We now are no longer interested in the value of the remainder but will consider numbers a and a' as equivalent if the remainder with division by a given number b is equal.
- ▶ Consider the clock:
 - ▶ It's now 3 o'clock
 - ▶ In 12 hours its 3 o'clock
 - ▶ Same in 24, 36, 48, ... hours.
 - ▶ 15:00 and 3:00 are shown the same.
 - ▶ In the following, we will express this as $3 \equiv 15 \pmod{12}$



Congruence Modulo n – Definition

Definition (Congruence modulo n)

For integer $n > 1$, two integers a and b are called **congruent modulo n** if $n \mid a - b$.

We write this as $a \equiv b \pmod{n}$.

Which of the following statements are true?

- ▶ $0 \equiv 5 \pmod{5}$
- ▶ $1 \equiv 6 \pmod{5}$
- ▶ $4 \equiv 14 \pmod{5}$
- ▶ $-8 \equiv 7 \pmod{5}$
- ▶ $2 \equiv -3 \pmod{5}$

Why is this the same concept as described in the clock example?!?

German: kongruent modulo n

Congruence Corresponds to Equal Remainders

Theorem

For integers a and b and integer $n > 1$ it holds that $a \equiv b \pmod{n}$ iff there are $q, q', r \in \mathbb{Z}$ with

$$\begin{aligned} a &= qn + r \\ b &= q'n + r. \end{aligned}$$

Proof sketch.

“ \Rightarrow ”: If $n \mid a - b$ then there is a $k \in \mathbb{Z}$ with $kn = a - b$.

As $n \neq 0$, by Euclid's lemma there are $q, q', r, r' \in \mathbb{Z}$ with $a = qn + r$ and $b = q'n + r'$, where $0 \leq r < |n|$ and $0 \leq r' < |n|$.

Together, we get that $kn = qn + r - (q'n + r')$, which is the case iff $kn + r' = (q - q')n + r$. By Euclid's lemma, quotients and remainders are unique, so in particular $r' = r$.

“ \Leftarrow ”: If we subtract the equations, we get $a - b = (q - q')n$, so $n \mid a - b$ and $a \equiv b \pmod{n}$.

Congruence Modulo n is an Equivalence Relation

Theorem

Congruence modulo n is an equivalence relation.

Proof sketch.

Reflexive: $a \equiv a \pmod{n}$ because every integer divides 0.

Symmetric: $a \equiv b \pmod{n}$ iff $n \mid a - b$ iff $n \mid b - a$ iff $b \equiv a \pmod{n}$.

Transitive: If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ then $n \mid a - b$ and $n \mid b - c$. Together, these imply that $n \mid a - b + b - c$. From $n \mid a - c$ we get $a \equiv c \pmod{n}$.

For modulus n , the equivalence class of a is

$$\bar{a}_n = \{ \dots, a - 2n, a - n, a, a + n, a + 2n, \dots \}.$$

Set \bar{a}_n is called the **congruence class** or **residue** of a modulo n .

German: Restklasse

Compatibility with Operations

Theorem

Congruence modulo n is **compatible with addition, subtraction, multiplication, translation, scaling and exponentiation**, i. e.

if $a \equiv b \pmod{n}$ and $a' \equiv b' \pmod{n}$ then

- ▶ $a + a' \equiv b + b' \pmod{n}$,
- ▶ $a - a' \equiv b - b' \pmod{n}$,
- ▶ $aa' \equiv bb' \pmod{n}$,
- ▶ $a + k \equiv b + k \pmod{n}$ for all $k \in \mathbb{Z}$,
- ▶ $ak \equiv bk \pmod{n}$ for all $k \in \mathbb{Z}$, and
- ▶ $a^k \equiv b^k \pmod{n}$ for all $k \in \mathbb{N}_0$.

Congruence modulo n is a so-called **congruence relation** (= equivalence relation compatible with operations).

German: kompatibel mit Addition, Subtraktion, Multiplikation,

Translation, Skalierung, Exponentiation; Kongruenzrelation

Summary

- ▶ **m divides n** (written $m \mid n$) if n is a multiple of m , i.e. there is an integer k with $n = mk$.
- ▶ Divisibility is compatible with multiplication and exponentiation.
- ▶ Divisibility over the natural numbers is a partial order.
- ▶ The **modulo operation** $a \bmod b$ corresponds to the remainder of Euclidean division.
- ▶ **Congruence modulo n** considers integers equivalent if they have with divisor n the same remainder.
- ▶ Congruence modulo n is an equivalence relation that is compatible with the arithmetic operations.