Discrete Mathematics in Computer Science

B7. Sets: Countability

Malte Helmert, Gabriele Röger

University of Basel

October 28, 2024

Countable Sets

Comparing Cardinality

- Two sets A and B have the same cardinality
  if their elements can be paired
  (i.e. there is a bijection from A to B).
- Set A has a strictly smaller cardinality than set B if
  - we can map distinct elements of A to distinct elements of B (i.e. there is an injective function from A to B), and
  - $|A| \neq |B|$.

Comparing Cardinality

- Two sets A and B have the same cardinality
  if their elements can be paired
  (i.e. there is a bijection from A to B).
- Set A has a strictly smaller cardinality than set B if
  - we can map distinct elements of A to distinct elements of B (i.e. there is an injective function from A to B), and
  - $|A| \neq |B|$.

- This clearly makes sense for finite sets.

Comparing Cardinality

- Two sets A and B have the same cardinality
  if their elements can be paired
  (i.e. there is a bijection from A to B).
- Set A has a strictly smaller cardinality than set B if
  - we can map distinct elements of A to distinct elements of B (i.e. there is an injective function from A to B), and
  - $|A| \neq |B|$.

- This clearly makes sense for finite sets.
- What about infinite sets?
  Do they even have different cardinalities?

Countable and Countably Infinite Sets

Definition (countably infinite and countable)

A set A is countably infinite if $|A| = |\mathbb{N}_0|$.

A set A is countable if $|A| \leq |\mathbb{N}_0|$.

A set is countable if it is finite or countably infinite.

Countable and Countably Infinite Sets

Definition (countably infinite and countable)

A set A is countably infinite if $|A| = |\mathbb{N}_0|$.

A set A is countable if $|A| \leq |\mathbb{N}_0|$.

A set is countable if it is finite or countably infinite.

- We can count the elements of a countable set one at a time.
- The objects are "discrete" (in contrast to "continuous").
- Discrete mathematics deals with all kinds of countable sets.

Set of Even Numbers

- $\mathit{even} = \{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}$
- Obviously: $\mathit{even} \subset \mathbb{N}_0$
- Intuitively, there are twice as many natural numbers as even numbers — no?
- Is $|\mathit{even}| < |\mathbb{N}_0|$?

Set of Even Numbers

Theorem (set of even numbers is countably infinite)

The set of all even natural numbers is countably infinite,
i. e. $|\{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}| = |\mathbb{N}_0|$.

Set of Even Numbers

Theorem (set of even numbers is countably infinite)

The set of all even natural numbers is countably infinite,
i. e. $|\{n \mid n \in \mathbb{N}_0 \text{ and } n \text{ is even}\}| = |\mathbb{N}_0|$.

Proof Sketch.

We can pair every even number $2n$ with natural number n. ☐

Set of Perfect Squares

Theorem (set of perfect squares is countably infinite)

The set of all perfect squares is countably infinite,
i. e. $|\{n^2 \mid n \in \mathbb{N}_0\}| = |\mathbb{N}_0|$.

Set of Perfect Squares

Theorem (set of perfect squares is countably infinite)

The set of all perfect squares is countably infinite,
i. e. $|\{n^2 \mid n \in \mathbb{N}_0\}| = |\mathbb{N}_0|$.

Proof Sketch.

We can pair every square number n^2 with natural number n. ☐

Subsets of Countable Sets are Countable

In general:

Theorem (subsets of countable sets are countable)

Let A be a countable set. Every set B with $B \subseteq A$ is countable.

Subsets of Countable Sets are Countable

In general:

Theorem (subsets of countable sets are countable)

Let A be a countable set. Every set B with $B \subseteq A$ is countable.

Proof.

Since A is countable there is an injective function f from A to $\mathbb{N}_0$. The restriction of f to B is an injective function from B to $\mathbb{N}_0$. ☐

Set of the Positive Rationals

Theorem (set of positive rationals is countably infinite)

Set $\mathbb{Q}_+ = \{n \mid n \in \mathbb{Q} \text{ and } n > 0\} = \{p/q \mid p, q \in \mathbb{N}_1\}$
is countably infinite.

Set of the Positive Rationals

Theorem (set of positive rationals is countably infinite)

Set $\mathbb{Q}_+ = \{n \mid n \in \mathbb{Q} \text{ and } n > 0\} = \{p/q \mid p, q \in \mathbb{N}_1\}$ *is countably infinite.*

Proof idea.

$$\begin{tikzcd}
\frac{1}{1}\ (0) \arrow[r] & \frac{1}{2}\ (1) \arrow[dl] & \frac{1}{3}\ (4) \arrow[r] & \frac{1}{4}\ (5) \arrow[dl] & \frac{1}{5}\ (10) \arrow[r] & {} \\
\frac{2}{1}\ (2) \arrow[d] & \frac{2}{2}\ (\cdot) \arrow[ur] & \frac{2}{3}\ (6) \arrow[dl] & \frac{2}{4}\ (\cdot) \arrow[ur] & \frac{2}{5} & \cdots \\
\frac{3}{1}\ (3) \arrow[ur] & \frac{3}{2}\ (7) \arrow[dl] & \frac{3}{3}\ (\cdot) \arrow[ur] & \frac{3}{4} & \frac{3}{5} & \cdots \\
\frac{4}{1}\ (8) \arrow[d] & \frac{4}{2}\ (\cdot) \arrow[ur] & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \cdots \\
\frac{5}{1}\ (9) \arrow[ur] & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

□

Union of Two Countable Sets is Countable

Theorem (union of two countable sets countable)

Let A and B be countable sets. Then $A \cup B$ is countable.

Proof sketch.

As A and B are countable there is an injective function f_A from A to $\mathbb{N}_0$, analogously f_B from B to $\mathbb{N}_0$.

We define function $f_{A\cup B}$ from $A \cup B$ to $\mathbb{N}_0$ as

$$f_{A\cup B}(e) = \begin{cases} 2f_A(e) & \text{if } e \in A \\ 2f_B(e) + 1 & \text{otherwise} \end{cases}$$

This $f_{A\cup B}$ is an injective function from $A \cup B$ to $\mathbb{N}_0$. □

Integers and Rationals

Theorem (sets of integers and rationals are countably infinite)

The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.

Without proof ($\leadsto$ exercises)

Union of More than Two Sets

Definition (arbitrary unions)

Let M be a set of sets. The union $\bigcup_{S \in M} S$ is the set with

$$x \in \bigcup_{S \in M} S \text{ iff exists } S \in M \text{ with } x \in S.$$

Countable Union of Countable Sets

Theorem

Let M be a countable set of countable sets.

Then $\bigcup_{S \in M} S$ is countable.

Proof sketch.

With $M = \{S_1, S_2, S_3, \dots\}$ (possibly finite) and each $S_i = \{x_{i1}, x_{i2}, \dots\}$ (possibly finite), we can use an analogous idea as for the countability of $\mathbb{Q}_+$ (skipping duplicates):

$$\begin{tikzcd}
S_1 : & x_{11} \arrow[r] & x_{12} \arrow[dl] & x_{13} \arrow[r] & x_{14} \arrow[dl] & x_{15} \arrow[r] & {} \\
S_2 : & x_{21} \arrow[d] & x_{22} \arrow[ur] & x_{23} \arrow[dl] & x_{24} \arrow[ur] & x_{25} & \cdots \\
S_3 : & x_{31} \arrow[ur] & x_{32} \arrow[dl] & x_{33} \arrow[ur] & x_{34} & x_{35} & \cdots \\
S_4 : & x_{41} \arrow[d] & x_{42} \arrow[ur] & x_{43} & x_{44} & x_{45} & \cdots \\
S_5 : & x_{51} \arrow[ur] & x_{52} & x_{53} & x_{54} & x_{55} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}$$

□

Set of all Binary Trees is Countable

Theorem (set of all binary trees is countable)

The set $B = \{b \mid b \text{ is a binary tree}\}$ is countable.

Proof.

For $n \in \mathbb{N}_0$ the set B_n of all binary trees with n leaves is finite.

With $M = \{B_i \mid i \in \mathbb{N}_0\}$ the set of all binary trees is $B = \bigcup_{B' \in M} B'$.

Since M is a countable set of countable sets, B is countable. ☐

And Now?

We have seen several countably infinite sets.

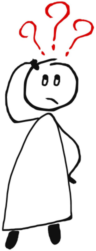
And Now?

We have seen several countably infinite sets.

What about our original questions?

- Do all infinite sets have the same cardinality?
- Are they all countably infinite?

Questions



Questions?

Summary

Summary

- A set is countable if it has at most cardinality $|\mathbb{N}_0|$.
- If a set is countable and infinite, it is countably infinite.
- Sets $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.

Summary

- A set is countable if it has at most cardinality $|\mathbb{N}_0|$.
- If a set is countable and infinite, it is countably infinite.
- Sets $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite.
- Every subset of a countable set is countable.
- Every countable union of countable sets is countable, in particular, the union of two countable sets is countable.